Identifying and interpreting the factors in factor models via sparsity: Different approaches

Online Appendix

Thomas Despois[1,2] and Catherine Doz[1,2]

[1]Paris School of Economics
[2]Université Paris 1 Panthéon-Sorbonne

A Proof of Theorem 1

Preliminary results and notations

We use the same set of assumptions as in Doz, Giannone, and Reichlin (2011). We suppose that the normalization condition is the same as in Doz, Giannone, and Reichlin (2011), which is innocuous, since F_t is defined up to a rotation matrix. In particular, the true value of the loading matrix Λ, satisfies $\Lambda'\Lambda = D$, where D is a diagonal $r \times r$ matrix whose terms go to infinity linearly with N.[1]

PCA notations

We denote:

- $\widehat{d}_1 \geqslant \cdots \geqslant \widehat{d}_n$ the ordered eigenvalues of $\frac{X'X}{T}$
- $\widehat{p}_1, \ldots, \widehat{p}_n$ an associated orthonormal set of eigenvectors
- $\widehat{P} = (\widehat{p}_1, \ldots, \widehat{p}_r)$
- $\widehat{D} = \text{diag}(\widehat{d}_1, \ldots, \widehat{d}_r)$

Other notations

- We use the same notations as in the description of the SPCA algorithm.
- For any matrix A, we take $\|A\| = \sqrt{\sigma_{max}(A'A)}$, where $\sigma_{max}(A'A)$ is the maximum eigenvalue of $A'A$. For any symmetric positive matrix A, we then have $\|A\| = \sigma_{max}(A)$.

[1]Here we drop the index 0 which Doz, Giannone, and Reichlin (2011) use for the true values of the parameters.

- For any matrix A, we say that $A = O_P(f(N))$ if $\|A\| = O_P(f(N))$.
- For any symmetric matrix A, we denote by $\sigma_{min}(A)$ the smallest eigenvalue of A.

To prove our theorem, we use properties which have been obtained by Doz, Giannone, and Reichlin (2011) under their set of assumptions or which can be easily derived from properties which they have stated. In particular, we use the following properties.

Preliminary properties (PCA properties)

i) $\widehat{D} = O_P(N)$

ii) $\widehat{D}^{-1} = O_P\left(\frac{1}{N}\right)$

iii) $\left(\frac{X'X}{T}\right)^{-1} = O_P(1)$

Proof.

i) Doz, Giannone, and Reichlin (2011) have shown that (see Lemma 2 in their Appendix):

$$\frac{1}{N}\left(\widehat{D} - D\right) = O\left(\frac{1}{N}\right) + O_P\left(\frac{1}{\sqrt{T}}\right)$$

We can thus write: $\widehat{D} = D + O(1) + O_P\left(\frac{N}{\sqrt{T}}\right)$.
As all the terms of D tend to infinity linearly with N, we thus obtain $\widehat{D} = O_P(N)$.

ii) Doz, Giannone, and Reichlin (2011) have also shown in the same lemma that:

$$N\left(\widehat{D}^{-1} - D^{-1}\right) = O\left(\frac{1}{N}\right) + O_P\left(\frac{1}{\sqrt{T}}\right)$$

We can thus write: $\widehat{D}^{-1} = D^{-1} + O\left(\frac{1}{N^2}\right) + O_P\left(\frac{1}{N\sqrt{T}}\right)$.
As all the terms of D tend to infinity linearly with N, we thus obtain $\widehat{D}^{-1} = O_P\left(\frac{1}{N}\right)$.

iii) If we denote $\widehat{\Sigma}_e = \frac{X'X}{T} - \widehat{\Lambda}\widehat{\Lambda}'$, Doz, Giannone, and Reichlin (2011) have shown that:[2]

$$\widehat{\sigma}_{ij,e} - \sigma_{ij,e} = O_P\left(\frac{1}{\sqrt{N}}\right) + O_P\left(\frac{1}{\sqrt{T}}\right) \quad \text{(A1)}$$

and that the result is uniform w.r.t. (i, j).

By Assumption (CR4) we know that $\sigma_{min}(\Sigma_e) = c > 0$.

By the Weyl theorem, we know that:

$$\sigma_{min}\left(\widehat{\Sigma}_e\right) \geqslant \sigma_{min}\left(\widehat{\Sigma}_e - \Sigma_e\right) + \sigma_{min}(\Sigma_e) \quad \text{(A2)}$$

[2] Σ_e is denoted Ψ_0 in Doz, Giannone, and Reichlin (2011) and $\widehat{\Sigma}_e$ is denoted $\widehat{\Psi}$.

Further:

$$\begin{aligned}
\sigma_{min}\left(\widehat{\Sigma}_e - \Sigma_e\right) &= \min_{x'x=1} x'\left(\widehat{\Sigma}_e - \Sigma_e\right) x \\
&= \min_{x'x=1} \sum_{i,j} x_i x_j \left(\widehat{\sigma}_{ij,e} - \sigma_{ij,e}\right) \\
&\leqslant \min_{x'x=1} \sum_{i,j} |x_i||x_j| \left|\widehat{\sigma}_{ij,e} - \sigma_{ij,e}\right| \\
&\leqslant \max_{i,j} \left|\widehat{\sigma}_{ij,e} - \sigma_{ij,e}\right| \min_{x'x=1} \sum_{i,j} |x_i||x_j| \\
&\leqslant \max_{i,j} \left|\widehat{\sigma}_{ij,e} - \sigma_{ij,e}\right| \text{ by the Cauchy-Schwarz inequality} \\
&= O_P\left(\frac{1}{\sqrt{N}}\right) + O_P\left(\frac{1}{\sqrt{T}}\right)
\end{aligned}$$

Using Equation (A2), we thus have: $\sigma_{min}\left(\widehat{\Sigma}_e\right) \geqslant c + O_P\left(\frac{1}{\sqrt{N}}\right) + O_P\left(\frac{1}{\sqrt{T}}\right)$.

As $\sigma_{min}\left(\frac{X'X}{T}\right) \geqslant \sigma_{min}\left(\widehat{\Sigma}_e\right)$, we also get $\sigma_{min}\left(\frac{X'X}{T}\right) \geqslant c + O_P\left(\frac{1}{\sqrt{N}}\right) + O_P\left(\frac{1}{\sqrt{T}}\right)$ so that $\sigma_{max}\left(\frac{X'X}{T}^{-1}\right) \leqslant \frac{1}{c} + O_P\left(\frac{1}{\sqrt{T}}\right)$ and $\left(\frac{X'X}{T}\right)^{-1} = O_P(1)$. □

Before proving our theorem, we now provide a few intermediary results concerning the algorithm, which are stated in the following properties.

Proposition 1. *At iteration k, $B^{(k)}$ can be written as*

$$B^{(k)} = \left(\frac{X'X}{T} + \kappa_2 I_N\right)^{-1}\left(\frac{X'X}{T}A^{(k)} - \frac{\kappa_1}{2}Z^{(k)}\right) \tag{A3}$$

where $Z^{(k)}$ is a $N \times r$ matrix whose general term satisfies $|z_{ij}^{(k)}| \leqslant 1$.

Proof. We know from Zou, Hastie, and Tibshirani (2006) (see the result recalled in our presentation of step 1 of the algorithm) that $B^{(k)}$ is obtained as $B^{(k)} = \left(b_1^{(k)}, \ldots, b_r^{(k)}\right)$ where

$$b_j^{(k)} = \arg\min_b\, (a_j^{(k)} - b)'\frac{X'X}{T}(a_j^{(k)} - b) + \kappa_1\|b\|_1 + \kappa_2\|b\|_2^2, \text{ for } j = 1, \ldots, r. \tag{A4}$$

For a given k, let us denote $f_j(b) = (a_j^{(k)} - b)'\frac{X'X}{T}(a_j^{(k)} - b) + \kappa_1\|b\|_1 + \kappa_2\|b\|_2^2$ and $b_j^{(k)} = \arg\min_b f_j(b)$. Standard convex analysis results (see Rockafellar, 1970, Part VI, Section 27) imply that 0 belongs to the subdifferential of f_j in $b_j^{(k)}$, denoted as $\partial f_j(b_j^{(k)})$.

Let us then calculate $\partial f_j(b)$ the subdifferential of f_j in b, for any b.

If we denote $g_j(b) = (a_j^{(k)} - b)'\frac{X'X}{T}(a_j^{(k)} - b) + \kappa_2\|b\|_2^2$, g_j is differentiable, with gradient

$$\nabla g_j(b) = -2\frac{X'X}{T}a_j^{(k)} + 2\frac{X'X}{T}b + 2\kappa_2 b.$$

For $b = (b_1, \ldots, b_n)'$ let us denote $h(b) = \|b\|_1 = \sum_{i=1}^n |b_i|$. If $z = (z_1, \ldots, z_n)'$, we know (see Rockafellar, 1970, Part V, Section 23) that z belongs to $\partial h(b)$, the subdifferential of h in b, if and only if:

$$\begin{cases} z_i = 1 & \text{if } b_i > 0 \\ z_i = -1 & \text{if } b_i < 0 \\ z_i \in [-1, 1] & \text{if } b_i = 0 \end{cases} \tag{A5}$$

Thus, the elements of $\partial f_j(b)$ can be written as:

$$-2\frac{X'X}{T}a_j^{(k)} + 2\frac{X'X}{T}b + 2\kappa_2 b + \kappa_1 z \tag{A6}$$

where z satisfies Equations (A5).

We then obtain that $b_j^{(k)} = \arg\min_b f_j(b)$ if and only if there exists a z_j such that:

$$-2\frac{X'X}{T}a_j^{(k)} + 2\frac{X'X}{T}b_j^{(k)} + 2\kappa_2 b_j^{(k)} + \kappa_1 z_j = 0 \tag{A7}$$

and

$$\begin{cases} z_{ij} = 1 & \text{if } b_{ij}^{(k)} > 0 \\ z_{ij} = -1 & \text{if } b_{ij}^{(k)} < 0 \\ z_{ij} \in [-1, 1] & \text{if } b_{ij}^{(k)} = 0 \end{cases} \tag{A8}$$

Solving Equation (A7) for $b_j^{(k)}$, we then get:

$$b_j^{(k)} = \left(\frac{X'X}{T} + \kappa_2 I_N\right)^{-1} \left(\frac{X'X}{T}a_j^{(k)} - \frac{\kappa_1}{2}z_j\right) \tag{A9}$$

If we denote $B^{(k)} = \left(b_1^{(k)}, \ldots, b_r^{(k)}\right)$, $A^{(k)} = \left(a_1^{(k)}, \ldots, a_r^{(k)}\right)$, and $Z^{(k)} = (z_1, \ldots, z_r)$, the result of Proposition 1 follows. □

Proposition 1 gives the general form of $B^{(k)}$ as a function of $A^{(k)}$ at step 1 of iteration k. We now study the general form of $A^{(k)}$, and state the following proposition.

Proposition 2. *Under standard Assumptions A1-A3 and Assumptions CR1-CR4 in Doz, Giannone, and Reichlin (2011), if* $\kappa_1 = O\left(\frac{1}{\sqrt{N}}\right)$*, the following result holds: for any* k*,* $A^{(k)}$ *can be written as* $A^{(k)} = \widehat{P}M_k + O_P\left(\frac{1}{N}\right)$ *where* M_k *is a* $r \times r$ *matrix which satisfies* $M_k'M_k = I_r + O_P\left(\frac{1}{N}\right)$*.*

The proof of Proposition 2 will be done by mathematical induction, and relies on the following lemma.

Lemma 1. *If we suppose that* $A^{(k)} = \widehat{P}M_k + O_P\left(\frac{1}{N}\right)$ *where* M_k *is a* $r \times r$ *matrix which satisfies* $M_k'M_k = I_r + O_P\left(\frac{1}{N}\right)$*, then:*

(i) $B^{(k)} = \widehat{P}M_k + O_P\left(\frac{1}{N}\right) - \frac{\kappa_1}{2}\left(\frac{X'X}{T} + \kappa_2 I_N\right)^{-1} Z^{(k)}$

(ii) $\frac{X'X}{T}B^{(k)} = \widehat{P}\widehat{D}M_k + O_P(1)$

Proof of Lemma 1.

i) Using Proposition 1, we know from Equation (A3) that

$$B^{(k)} = \left(\frac{X'X}{T} + \kappa_2 I_N\right)^{-1} \left(\frac{X'X}{T} A^{(k)} - \frac{\kappa_1}{2} Z^{(k)}\right)$$

with $Z^{(k)}$ is a $N \times r$ matrix whose general term satisfies $|z_{ij}^{(k)}| \leqslant 1$.

If $A^{(k)} = \widehat{P} M_k + O_P\left(\frac{1}{N}\right)$, this can be written as

$$B^{(k)} = \left(\frac{X'X}{T} + \kappa_2 I_N\right)^{-1} \frac{X'X}{T} \left(\widehat{P} M_k + O_P\left(\frac{1}{N}\right)\right) - \frac{\kappa_1}{2} \left(\frac{X'X}{T} + \kappa_2 I_N\right)^{-1} Z^{(k)} \tag{A10}$$

or equivalently

$$B^{(k)} = \left(\frac{X'X}{T} + \kappa_2 I_N\right)^{-1} \frac{X'X}{T} \widehat{P} M_k + \left(\frac{X'X}{T} + \kappa_2 I_N\right)^{-1} \frac{X'X}{T} \times O_P\left(\frac{1}{N}\right) - \frac{\kappa_1}{2} \left(\frac{X'X}{T} + \kappa_2 I_N\right)^{-1} Z^{(k)} \tag{A11}$$

Let us now study the first two terms of (A11).

We know that the columns of $\widehat{P}$ are eigenvectors of $\frac{X'X}{T}$, associated to the eigenvalues contained in the diagonal matrix $\widehat{D}$. It follows that the columns of $\widehat{P}$ are also eigenvectors of $\frac{X'X}{T} + \kappa_2 I_N$ associated to the eigenvalues contained in the matrix $\widehat{D} + \kappa_2 I_r$, and eigenvectors of $(\frac{X'X}{T} + \kappa_2 I_N)^{-1}$ associated to the eigenvalues contained in the matrix $(\widehat{D} + \kappa_2 I_r)^{-1}$.

We thus get:

$$\left(\frac{X'X}{T} + \kappa_2 I_N\right)^{-1} \frac{X'X}{T} \widehat{P} M_k = \widehat{P} (\widehat{D} + \kappa_2 I_r)^{-1} \widehat{D} M_k \tag{A12}$$

Further, the diagonal terms of $(\widehat{D} + \kappa_2 I_r)^{-1} \widehat{D}$ are $\frac{\widehat{d}_i}{\widehat{d}_i + \kappa_2}$ for $i = 1, \ldots, r$, with

$$\frac{\widehat{d}_i}{\widehat{d}_i + \kappa_2} = \frac{1}{1 + \frac{\kappa_2}{\widehat{d}_i}} \sim 1 - \frac{\kappa_2}{\widehat{d}_i} = 1 + O_P\left(\frac{1}{N}\right) \tag{A13}$$

so that $(\widehat{D} + \kappa_2 I_r)^{-1} \widehat{D} = I_r + O_P\left(\frac{1}{N}\right)$.

As $\widehat{P}' \widehat{P} = I_r$, we have $\widehat{P} = O_P(1)$. As $M_k' M_k = I_r + O_P\left(\frac{1}{N}\right)$, we also have $M_k = O_P(1)$. We can then write:

$$\widehat{P} (\widehat{D} + \kappa_2 I_r)^{-1} \widehat{D} M_k = \widehat{P} (I_r + O_P\left(\frac{1}{N}\right)) M_k = \widehat{P} M_k + O_P\left(\frac{1}{N}\right) \tag{A14}$$

Using (A12), we thus get:

$$\left(\frac{X'X}{T} + \kappa_2 I_N\right)^{-1} \frac{X'X}{T} \widehat{P} M_k = \widehat{P} M_k + O_P\left(\frac{1}{N}\right). \tag{A15}$$

Turning to the second term of (A11), we notice that the eigenvalues of $\left(\frac{X'X}{T} + \kappa_2 I_N\right)^{-1} \frac{X'X}{T}$ are $\frac{\widehat{d}_i}{\widehat{d}_i + \kappa_2}, i = 1, \ldots, n$, which are all smaller than 1.

We thus get: $\left\| \left(\frac{X'X}{T} + \kappa_2 I_N\right)^{-1} \frac{X'X}{T} \right\| \leqslant 1$, so that this second term is $O_P\left(\frac{1}{N}\right)$.

Part (i) of the lemma then follows.

ii) It follows from (i) that:

$$\frac{X'X}{T}B^{(k)} = \frac{X'X}{T}\left(\widehat{P}M_k + O_P\left(\frac{1}{N}\right) - \frac{\kappa_1}{2}\left(\frac{X'X}{T} + \kappa_2 I_N\right)^{-1} Z^{(k)}\right) \tag{A16}$$

$$= \widehat{P}\widehat{D}M_k + \frac{X'X}{T} \times O_P\left(\frac{1}{N}\right) - \frac{\kappa_1}{2}\frac{X'X}{T}\left(\frac{X'X}{T} + \kappa_2 I_N\right)^{-1} Z^{(k)} \tag{A17}$$

As $\frac{X'X}{T} = O_P(N)$, we thus get:

$$\frac{X'X}{T}B^{(k)} = \widehat{P}\widehat{D}M_k + O_P(1) - \frac{\kappa_1}{2}\frac{X'X}{T}\left(\frac{X'X}{T} + \kappa_2 I_N\right)^{-1} Z^{(k)} \tag{A18}$$

Further:

- $\frac{X'X}{T}\left(\frac{X'X}{T} + \kappa_2 I_N\right)^{-1}$ has the same eigenvalues than $\left(\frac{X'X}{T} + \kappa_2 I_N\right)^{-1}\frac{X'X}{T}$ so that

$$\left\|\frac{X'X}{T}\left(\frac{X'X}{T} + \kappa_2 I_N\right)^{-1}\right\| = \left\|\left(\frac{X'X}{T} + \kappa_2 I_N\right)^{-1}\frac{X'X}{T}\right\| \leqslant 1$$

- $\left\|Z^{(k)}\right\|^2 \leqslant \operatorname{tr}\left(Z^{(k)\prime}Z^{(k)}\right) = \sum_{j=1}^{r}\sum_{i=1}^{N}\left(z_{ij}^{(k)}\right)^2 \leqslant rN$ so that $\left\|Z^{(k)}\right\| = O_P(\sqrt{N})$.

It then follows that:

$$\left\|\frac{X'X}{T}\left(\frac{X'X}{T} + \kappa_2 I_N\right)^{-1} Z^{(k)}\right\| \leqslant \left\|\frac{X'X}{T}\left(\frac{X'X}{T} + \kappa_2 I_N\right)^{-1}\right\|\left\|Z^{(k)}\right\| = O_p(\sqrt{N})$$

As $\kappa_1 = O\left(\frac{1}{\sqrt{N}}\right)$, part (ii) of the lemma follows. □

Proof of Proposition 2.

We prove Proposition 2 by mathematical induction.

For $k = 1$, we have $A^{(1)} = \widehat{P}$ and the result is true with $M_1 = I_r$.

If we suppose that the result is true for k, then Lemma 1 applies.

Besides, we know that $A^{(k+1)}$ is obtained from step 2 of iteration k in the following way:

if the SVD of $\frac{X'X}{T}B^{(k)}$ is written as $\frac{X'X}{T}B^{(k)} = U_k\Delta_k V_k'$, then $A^{(k+1)} = U_k V_k'$.

Further, the SVD of $\frac{X'X}{T}B$ is obtained in the following way:

- V_k is a $r \times r$ matrix whose columns are normalized eigenvectors associated to the r eigenvalues of $B^{(k)\prime}\left(\frac{X'X}{T}\right)^2 B^{(k)}$, so that $V_k'V_k = I_r$,

- Δ_k^2 is the diagonal $r \times r$ matrix whose diagonal terms are the associated eigenvalues
- U_k is the $N \times r$ matrix defined by $U_k = \frac{X'X}{T} B^{(k)} V_k \Delta_k^{-1}$, so that $U_k' U_k = I_r$ and

$$\frac{X'X}{T} B^{(k)} = U_k \Delta_k V_k'$$

From result (ii) of Lemma 1, we know that:

$$\frac{X'X}{T} B^{(k)} = \widehat{P}\widehat{D} M_k + O_P(1) \tag{A19}$$

Given the fact that $\widehat{P}'\widehat{P} = I_r$ and that $\widehat{D} = O_P(N)$, we first get

$$\begin{aligned} B^{(k)\prime} \left(\frac{X'X}{T}\right)^2 B^{(k)} &= (\widehat{P}\widehat{D} M_k + O_P(1))'(\widehat{P}\widehat{D} M_k + O_P(1)) \\ &= M_k' \widehat{D}^2 M_k + O_P(N) \end{aligned} \tag{A20}$$

so that

$$V_k \Delta_k^2 V_k' = M_k' \widehat{D}^2 M_k + O_P(N)$$

where V_k is a $r \times r$ matrix and $V_k' V_k = I_r$.

Thus Δ_k^2, the diagonal matrix which contains the eigenvalues of $B^{(k)\prime} \left(\frac{X'X}{T}\right)^2 B^{(k)}$, has terms which all diverge linearly with N^2, like the terms of $\widehat{D}^2$, and Δ_k has terms which all diverge linearly with N. We also get:

$$\begin{aligned} \Delta_k^2 &= V_k' M_k' \widehat{D}^2 M_k V_k + O_P(N) \\ \text{so that } I_r &= \Delta_k^{-1} \left(V_k' M_k' \widehat{D}^2 M_k V_k + O_P(N)\right) \Delta_k^{-1} \\ \text{and } I_r &= \Delta_k^{-1} V_k' M_k' \widehat{D}^2 M_k V_k \Delta_k^{-1} + O_P\left(\frac{1}{N}\right) \end{aligned} \tag{A21}$$

Now, we have seen that U_k is defined by:

$$U_k = \frac{X'X}{T} B^{(k)} V_k \Delta_k^{-1} \tag{A22}$$

As we know that $A^{(k+1)} = U_k V_k'$, we have $A^{(k+1)\prime} A^{(k+1)} = I_r$ by construction.

Using (A22), we can write $A^{(k+1)}$ as:

$$A^{(k+1)} = \frac{X'X}{T} B^{(k)} V_k \Delta_k^{-1} V_k'$$

Using Equation (A19), and the fact that $\Delta_k^{-1} = O_P\left(\frac{1}{N}\right)$, we then obtain:

$$\begin{aligned} A^{(k+1)} &= \left(\widehat{P}\widehat{D} M_k + O_P(1)\right) V_k \Delta_k^{-1} V_k' \\ &= \widehat{P}\widehat{D} M_k V_k \Delta_k^{-1} V_k' + O_P\left(\frac{1}{N}\right). \end{aligned}$$

If we denote $M_{k+1} = \widehat{D} M_k V_k \Delta_k^{-1} V_k'$, we have $A^{(k+1)} = \widehat{P} M_{k+1} + O_P\left(\frac{1}{N}\right)$.

As $A^{(k+1)\prime}A^{(k+1)} = I_r$, we get:

$$\left(\widehat{P}M_{k+1} + O_P\left(\frac{1}{N}\right)\right)'\left(\widehat{P}M_{k+1} + O_P\left(\frac{1}{N}\right)\right) = I_r \tag{A23}$$

As $\widehat{P}'\widehat{P} = I_r$ we know that $\widehat{P} = O_P(1)$.

Further, $M_{k+1} = \widehat{D}M_kV_k\Delta_k^{-1}V_k'$, with:

- $\widehat{D} = O_P(N)$
- $\Delta_k^{-1} = O_P\left(\frac{1}{N}\right)$
- $V_k'V_k = I_r$ so that $V_k = O_P(1)$
- $M_k'M_k = I_r + O_P\left(\frac{1}{N}\right)$ by assumption, so that $M_k = O_P(1)$.

We thus get: $M_{k+1} = O_P(1)$.

It then follows from Equation (A23) that

$$M_{k+1}'M_{k+1} + O_P\left(\frac{1}{N}\right) = I_r \tag{A24}$$

which completes the proof of Proposition 2. □

We are now ready to prove our consistency theorem for the estimated factors.

Proof of Theorem 1.

We suppose that the algorithm converges at iteration k and we construct the estimated loadings and factors as mentioned in the description of the algorithm. More precisely, we use the following notations: $\widetilde{B} = B^{(k)}$, $\widetilde{\Delta} = \text{diag}\left(\widetilde{B}'\frac{X'X}{T}\widetilde{B}\right)$, and $\widetilde{F}_t = \widetilde{\Delta}^{-1/2}\widetilde{B}'x_t$.

Using results (i) and (ii) of Lemma 1, and using the fact that $\widehat{D} = O_P(N)$, $\widehat{P} = O_P(1)$, and $M_k = O_P(1)$, we get that:

$$\begin{aligned}
\widetilde{B}'\frac{X'X}{T}\widetilde{B} &= \left(\widehat{P}M_k + O_P\left(\frac{1}{N}\right) - \frac{\kappa_1}{2}\left(\frac{X'X}{T} + \kappa_2 I_N\right)^{-1} Z^{(k)}\right)'\left(\widehat{P}\widehat{D}M_k + O_P(1)\right) \\
&= M_k'\widehat{D}M_k + O_P(1) - \frac{\kappa_1}{2}Z^{(k)\prime}\left(\frac{X'X}{T} + \kappa_2 I_N\right)^{-1}\widehat{P}\widehat{D}M_k - \frac{\kappa_1}{2}Z^{(k)\prime}\left(\frac{X'X}{T} + \kappa_2 I_N\right)^{-1} \times O_P(1) \\
&= M_k'\widehat{D}M_k + O_P(1) - \frac{\kappa_1}{2}Z^{(k)\prime}\widehat{P}\left(\widehat{D} + \kappa_2 I_N\right)^{-1}\widehat{D}M_k - \frac{\kappa_1}{2}Z^{(k)\prime}\left(\frac{X'X}{T} + \kappa_2 I_N\right)^{-1} \times O_P(1)
\end{aligned}$$

We have seen that $Z^{(k)} = O_P(\sqrt{N})$. As we assume that $\kappa_1 = O\left(\frac{1}{\sqrt{N}}\right)$, we get $\kappa_1 Z^{(k)} = O_P(1)$.

We also have seen that $\left(\widehat{D} + \kappa_2 I_N\right)^{-1}\widehat{D} \leqslant I_r$ so that $\left(\widehat{D} + \kappa_2 I_N\right)^{-1}\widehat{D} = O_P(1)$.
Further, $\left(\frac{X'X}{T} + \kappa_2 I_N\right)^{-1} \leqslant \left(\frac{X'X}{T}\right)^{-1} = O_P(1)$. We thus get:

$$\widetilde{B}'\frac{X'X}{T}\widetilde{B} = M_k'\widehat{D}M_k + O_P(1)$$

If we denote $M_k = \left(m_1^{(k)}, \ldots, m_r^{(k)}\right)$, the diagonal terms of $M_k' \widehat{D} M_k$ are

$$m_i^{(k)\prime} \widehat{D} m_i^{(k)} = \sum_{j=1}^{r} \widehat{d}_j \left(m_{ij}^{(k)}\right)^2$$

As $M_k' M_k = I_r + O_P\left(\frac{1}{N}\right)$, we have $\sum_{j=1}^{r} \widehat{d}_j \left(m_{ij}^{(k)}\right)^2 \in [\widehat{d}_r + O_P(1), \widehat{d}_1 + O_P(1)]$ for $i = 1, \ldots, r$ so that all the terms of $\widetilde{\Delta}$ diverge linearly with N like the terms of $\widehat{D}$. We thus have:

$$\widetilde{\Delta}^{-1/2} = O_P\left(\frac{1}{\sqrt{N}}\right)$$

As $\widetilde{F}_t = \widetilde{\Delta}^{-1/2} \widetilde{B}' x_t$, we can write, using result (i) of Lemma 1:

$$\widetilde{F}_t = \widetilde{\Delta}^{-1/2} \left[M_k' \widehat{P}' + O_P\left(\frac{1}{N}\right) - \frac{\kappa_1}{2} Z^{(k)\prime} \left(\frac{X'X}{T} + \kappa_2 I_N\right)^{-1} \right] x_t$$

and we know that the $O_P\left(\frac{1}{N}\right)$ term comes from Equation (A13) and is the same one for any t.

Using the fact that $\widehat{F}_t = \widehat{D}^{-1/2} \widehat{P}' x_t$, we then get:

$$\widetilde{F}_t = \widetilde{\Delta}^{-1/2} M_k' \widehat{D}^{1/2} \widehat{F}_t + O_P\left(\frac{1}{N}\right) - \widetilde{\Delta}^{-1/2} \frac{\kappa_1}{2} Z^{(k)\prime} \left(\frac{X'X}{T} + \kappa_2 I_N\right)^{-1} x_t \tag{A25}$$

If we denote $M = \widetilde{\Delta}^{-1/2} M_k' \widehat{D}^{1/2}$, it is clear that M is invertible and that $M = O_P(1)$.

Let us now show that $\frac{1}{T} \sum_{t=1}^{T} \|\widetilde{F}_t - M \widehat{F}_t\|_2^2 = O_P\left(\frac{1}{N}\right)$.

We can write:

$$\begin{aligned} &\frac{1}{T} \sum_{t=1}^{T} \|\widetilde{F}_t - M \widehat{F}_t\|_2^2 \\ =& \frac{1}{T} \sum_{t=1}^{T} \left(O_P\left(\frac{1}{N}\right) - \widetilde{\Delta}^{-1/2} \frac{\kappa_1}{2} Z^{(k)\prime} \left(\frac{X'X}{T} + \kappa_2 I_N\right)^{-1} x_t \right)' \left(O_P\left(\frac{1}{N}\right) - \widetilde{\Delta}^{-1/2} \frac{\kappa_1}{2} Z^{(k)\prime} \left(\frac{X'X}{T} + \kappa_2 I_N\right)^{-1} x_t \right) \end{aligned}$$

We have seen that:

. the $O_P\left(\frac{1}{N}\right)$ term does not depend on t

. $Z^{(k)} = O_P(\sqrt{N})$

. $\widetilde{\Delta}^{-1/2} = O_P\left(\frac{1}{\sqrt{N}}\right)$

. $\left(\frac{X'X}{T} + \kappa_2 I_N\right)^{-1} \leqslant \left(\frac{X'X}{T}\right)^{-1} = O_P(1)$

We can decompose $\frac{1}{T} \sum_{t=1}^{T} \|\widetilde{F}_t - M \widehat{F}_t\|_2^2$ into three terms that we study separately.

- The first term has the form $\frac{1}{T} \sum_{t=1}^{T} O_P\left(\frac{1}{N}\right) \times O_P\left(\frac{1}{N}\right)$ and the $O_P\left(\frac{1}{N}\right)$ term does not depend on t, so this term is $O_P\left(\frac{1}{N^2}\right)$.

- The second term is

$$2 \times O_P\left(\frac{1}{N}\right) \times \frac{1}{T}\sum_{t=1}^{T} \widetilde{\Delta}^{-1/2}\frac{\kappa_1}{2} Z^{(k)\prime}\left(\frac{X'X}{T} + \kappa_2 I_N\right) x_t \tag{A26}$$

which can be also written as

$$\kappa_1 \times O_P\left(\frac{1}{N}\right) \times \widetilde{\Delta}^{-1/2} Z^{(k)\prime}\left(\frac{X'X}{T} + \kappa_2 I_N\right) \frac{1}{T}\sum_{t=1}^{T} x_t \tag{A27}$$

Since $Z^{(k)} = O_P(\sqrt{N}), \widetilde{\Delta}^{-1/2} = O_P\left(\frac{1}{\sqrt{N}}\right), \left(\frac{X'X}{T} + \kappa_2 I_N\right)^{-1} = O_P(1)$, we know that:

$$\kappa_1 \times O_P\left(\frac{1}{N}\right) \times \widetilde{\Delta}^{-1/2} Z^{(k)\prime}\left(\frac{X'X}{T} + \kappa_2 I_N\right) = O_P\left(\frac{1}{N\sqrt{N}}\right).$$

Now $\frac{1}{T}\sum_{t=1}^{T} x_t = \Lambda \frac{1}{T}\sum_{t=1}^{T} F_t + \frac{1}{T}\sum_{t=1}^{T} e_t$, with:

- $\Lambda = O(\sqrt{N})$
- $\frac{1}{T}\sum_{t=1}^{T} F_t = O_P\left(\frac{1}{\sqrt{T}}\right)$ since (F_t) is a stationary process
- $\frac{1}{T}\sum_{t=1}^{T} e_t = O_P\left(\frac{\sqrt{N}}{\sqrt{T}}\right)$ by Assumption (CR3) since

$$E\left(\left\|\frac{1}{\sqrt{T}}\sum_{t=1}^{T} e_t\right\|^2\right) = E\left(\frac{1}{T}\sum_{t=1}^{T}\sum_{h\in\mathbb{Z}} e_t' e_{t-h}\right) = \frac{1}{T}\sum_{t=1}^{T}\sum_{h\in\mathbb{Z}} \text{tr } E\left(e_t e_{t-h}'\right) \leqslant N \sum_{h\in\mathbb{Z}} \|E(e_t e_{t-h}')\| = O(N).$$

We thus get that $\frac{1}{T}\sum_{t=1}^{T} x_t = O_P\left(\frac{\sqrt{N}}{\sqrt{T}}\right)$ and, using (A27), we obtain that (A26) is $O_P\left(\frac{1}{N\sqrt{T}}\right)$.

- The third term is

$$\frac{\kappa_1^2}{4}\frac{1}{T}\sum_{t=1}^{T} x_t' \left(\frac{X'X}{T} + \kappa_2 I_N\right)^{-1} Z^{(k)}\widetilde{\Delta}^{-1} Z^{(k)\prime}\left(\frac{X'X}{T} + \kappa_2 I_N\right)^{-1} x_t \tag{A28}$$

which can also be written as

$$\begin{aligned}
&\frac{\kappa_1^2}{4}\frac{1}{T}\sum_{t=1}^{T} \text{tr}\left[\widetilde{\Delta}^{-1/2} Z^{(k)\prime}\left(\frac{X'X}{T} + \kappa_2 I_N\right)^{-1} x_t x_t' \left(\frac{X'X}{T} + \kappa_2 I_N\right)^{-1} Z^{(k)}\widetilde{\Delta}^{-1/2}\right] \\
=&\frac{\kappa_1^2}{4}\text{tr}\left[\widetilde{\Delta}^{-1/2} Z^{(k)\prime}\left(\frac{X'X}{T} + \kappa_2 I_N\right)^{-1}\left(\frac{1}{T}\sum_{t=1}^{T} x_t x_t'\right)\left(\frac{X'X}{T} + \kappa_2 I_N\right)^{-1} Z^{(k)}\widetilde{\Delta}^{-1/2}\right] \\
=&\frac{\kappa_1^2}{4}\text{tr}\left[\widetilde{\Delta}^{-1/2} Z^{(k)\prime}\left(\frac{X'X}{T} + \kappa_2 I_N\right)^{-1}\left(\frac{X'X}{T}\right)\left(\frac{X'X}{T} + \kappa_2 I_N\right)^{-1} Z^{(k)}\widetilde{\Delta}^{-1/2}\right]
\end{aligned} \tag{A29}$$

Now, we can write

$$\left(\frac{X'X}{T} + \kappa_2 I_N\right)^{-1/2}\left(\frac{X'X}{T}\right)\left(\frac{X'X}{T} + \kappa_2 I_N\right)^{-1/2} \leqslant I_N$$

so that

$$\left(\frac{X'X}{T} + \kappa_2 I_N\right)^{-1}\left(\frac{X'X}{T}\right)\left(\frac{X'X}{T} + \kappa_2 I_N\right)^{-1} \leqslant \left(\frac{X'X}{T} + \kappa_2 I_N\right)^{-1} \leqslant \left(\frac{X'X}{T}\right)^{-1} = O_P(1)$$

Since $\kappa_1 = O\left(\frac{1}{\sqrt{N}}\right)$, $Z^{(k)\prime}Z^{(k)} = O_P(N)$, and $\widetilde{\Delta}^{-1/2} = O_P\left(\frac{1}{\sqrt{N}}\right)$, we then get

$$\frac{\kappa_1^2}{4}\widetilde{\Delta}^{-1/2}Z^{(k)\prime}\left(\frac{X'X}{T}+\kappa_2 I_N\right)^{-1}\left(\frac{X'X}{T}\right)\left(\frac{X'X}{T}+\kappa_2 I_N\right)^{-1}Z^{(k)}\widetilde{\Delta}^{-1/2} = O_P\left(\frac{1}{N}\right)$$

As this matrix is $r \times r$, its trace is also $O_P\left(\frac{1}{N}\right)$, so that (A29) is $O_P\left(\frac{1}{N}\right)$.

It then follows that the summation of the three terms of our decomposition is also $O_P\left(\frac{1}{N}\right)$ i.e. that

$$\frac{1}{T}\sum_{t=1}^{T}\|\widetilde{F}_t - M\widehat{F}_t\|_2^2 = O_P\left(\frac{1}{N}\right).$$

$\square$

B Supplementary material for the application to international business cycles

B.1 Data

The regional definitions follow Kose, Otrok, and Whiteman (2003). The Developing Asia region is renamed Emerging Asia.

- North America: Canada, Mexico, USA.
- Latin America: Argentina, Bolivia, Brazil, Chile, Colombia, Costa Rica, Dominican Republic, Ecuador, El Salvador, Guatemala, Honduras, Jamaica, Panama, Paraguay, Peru, Trinidad and Tobago, Uruguay, Venezuela.
- Europe: Austria, Belgium, Denmark, Finland, France, Germany, Greece, Iceland, Ireland, Italy, Luxembourg, Netherlands, Norway, Portugal, Spain, Sweden, Switzerland, United Kingdom.
- Africa: Cameroon, Côte d'Ivoire, Kenya, Morocco, Senegal, South Africa, Zimbabwe.
- Emerging Asia: Bangladesh, India, Indonesia, Pakistan, Philippines, Sri Lanka.
- Developed Asia: Hong Kong SAR, Japan, Korea, Malaysia, Singapore, Thailand.
- Oceania: Australia, New Zealand.

B.2 Additional results

Table B1: Selected number of factors (international business cycles).

Criterion	IC_{p1}	IC_{p2}	IC_{p3}	PC_{p1}	PC_{p2}	PC_{p3}
r*	1	1	4	2	1	4

Note. Number of factors selected by each of the six main Bai and Ng (2002) information criteria, allowing for at most four factors.

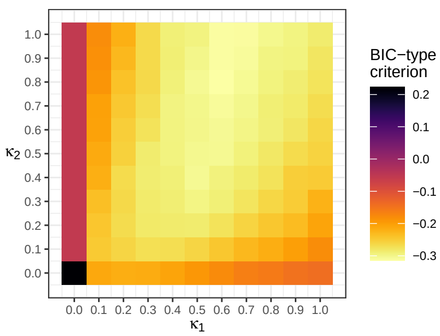


Figure B1: Grid search of the SPCA hyperparameters κ_1 and κ_2 (international business cycles).

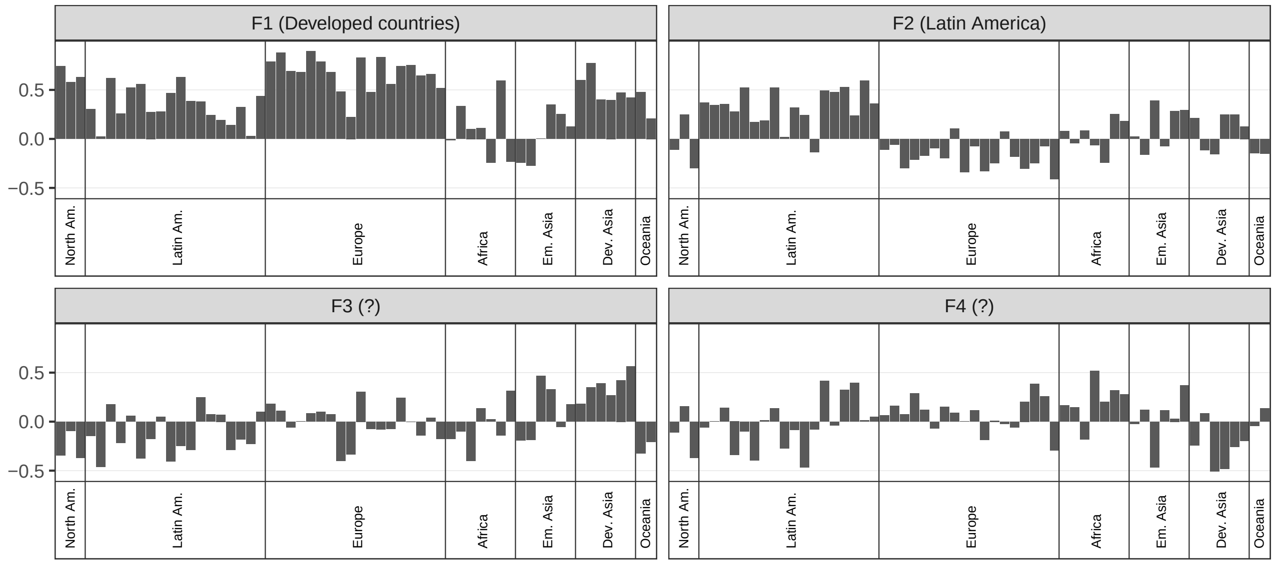


Figure B2: Factor loadings estimated with PCA (international business cycles).

Note. The interpretable factors are labeled with their economic interpretation, and the uninterpretable factors with a question mark. Since F and Λ are identified up to a column sign change, for each factor, we impose the largest loading in absolute value to be positive.

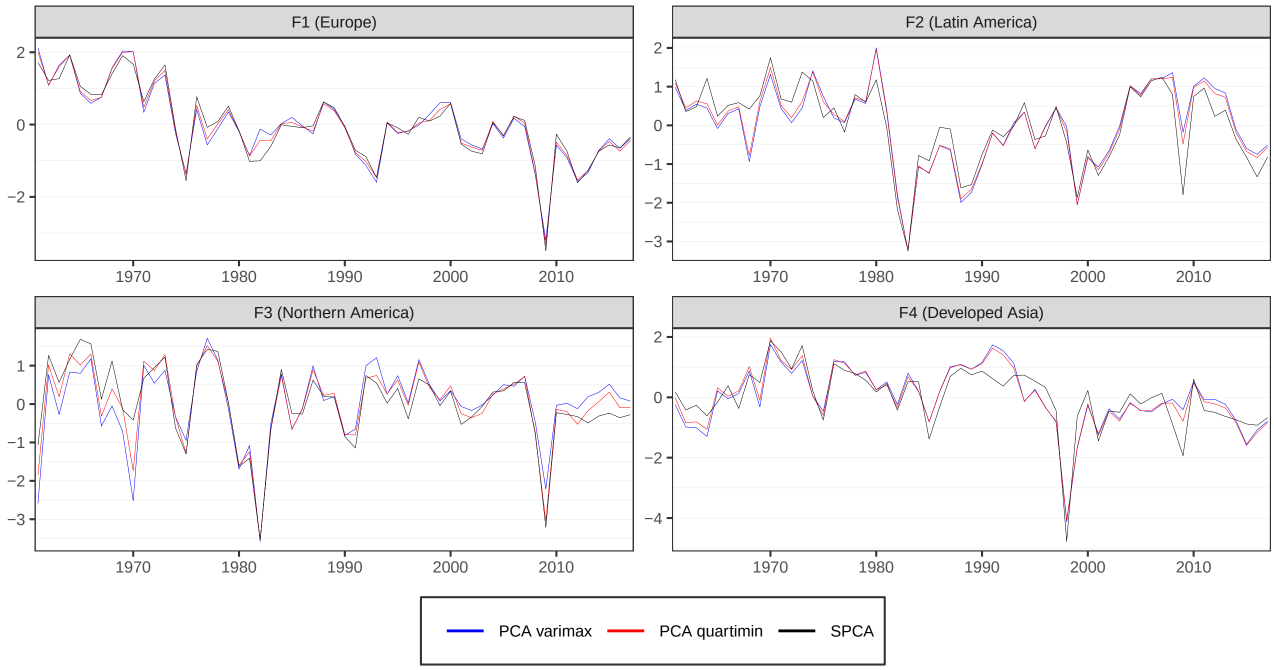


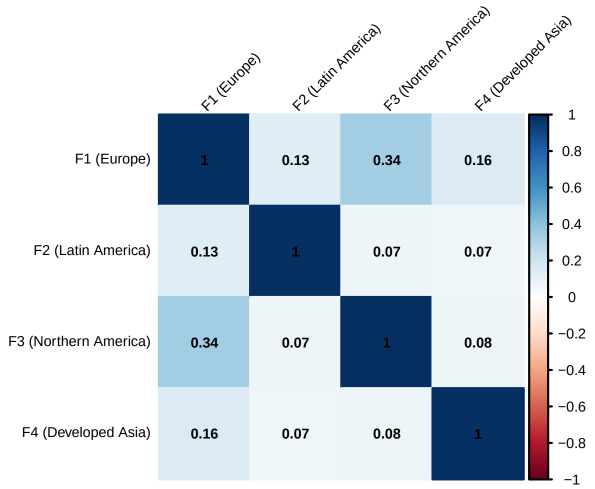
Figure B3: Factors estimated with PCA varimax, PCA quartimin, and SPCA (international business cycles).

Table B2: Correlations between the estimated factors (international business cycles).

Factors	PCA varimax vs. PCA quartimin	PCA varimax vs. SPCA	PCA quartimin vs. SPCA
F1 (Europe)	1.00	0.97	0.99
F2 (Latin America)	1.00	0.90	0.93
F3 (Northern America)	0.96	0.85	0.95
F4 (Developed Asia)	0.99	0.87	0.90

Table B3: Correlations between the estimated factors and the real GDP growth rates of some representative countries (international business cycles).

Pairs	PCA varimax	PCA quartimin	SPCA
F1 (Europe) vs. France	0.91	0.92	0.94
F2 (Latin America) vs. Brazil	0.47	0.52	0.66
F3 (Northern America) vs. USA	0.69	0.79	0.85
F4 (Developed Asia) vs. Malaysia	0.65	0.67	0.82



(a) PCA quartimin

(b) SPCA

Figure B4: Correlation matrices of the estimated factors (international business cycles).

Note. The correlation matrices of the factors estimated with PCA and PCA varimax are not displayed since they are simply the identity matrix, by construction.

Table B4: Variance decomposition with the factors estimated by PCA varimax (international business cycles).

Country	F1 (Europe)	F2 (Latin Am.)	F3 (Northern Am.)	F4 (Dev. Asia)	Common.
Canada	39.3	0.8	30.2	0.5	70.7
Mexico	22.1	18.2	3.0	0.2	43.5
USA	26.5	4.0	46.9	0.0	77.5
Argentina	1.5	15.4	6.9	2.0	25.7
Bolivia	2.6	15.8	12.5	3.0	33.9
Brazil	26.5	21.9	0.0	8.3	56.7
Chile	0.1	4.0	22.3	4.9	31.2
Colombia	8.7	26.8	4.8	16.3	56.6
Costa Rica	8.6	3.3	50.4	2.7	65.0
Dominican Republic	2.8	7.0	4.8	0.0	14.5
Ecuador	1.7	31.7	0.0	4.2	37.6
El Salvador	8.1	0.7	37.4	0.0	46.2
Guatemala	17.1	18.5	20.5	1.7	57.8
Honduras	1.1	2.6	40.9	6.9	51.5
Jamaica	16.7	1.4	0.0	5.5	23.6
Panama	3.3	40.8	4.3	0.1	48.6
Paraguay	0.1	18.3	0.4	8.8	27.6
Peru	0.0	46.4	0.6	2.5	49.5
Trinidad and Tobago	8.5	22.9	0.0	4.4	35.8
Uruguay	3.8	32.7	3.9	0.7	41.0
Venezuela	9.8	16.9	0.4	6.7	33.7
Austria	63.4	0.5	0.6	3.5	67.9
Belgium	77.4	3.5	1.0	1.4	83.3
Denmark	53.4	0.4	4.0	0.5	58.3
Finland	57.7	0.7	0.0	1.7	60.1
France	83.1	0.7	2.0	0.7	86.5
Germany	55.8	0.2	5.0	5.0	66.0
Greece	53.5	0.1	0.4	0.0	53.9
Iceland	13.2	9.8	15.1	3.6	41.7
Ireland	6.7	3.2	8.2	9.9	27.9
Italy	73.0	1.1	0.1	6.9	81.0
Luxembourg	22.8	6.1	9.4	0.1	38.3
Netherlands	68.1	0.0	9.3	0.0	77.4
Norway	21.7	3.7	6.4	1.0	32.8
Portugal	56.8	0.4	0.9	7.2	65.4
Spain	69.2	0.0	0.9	1.1	71.2
Sweden	55.0	1.3	0.2	9.4	66.0
Switzerland	48.5	3.2	0.0	0.1	51.8
United Kingdom	24.9	10.8	20.5	0.0	56.2
Cameroon	0.1	2.9	0.0	3.7	6.7
Côte d'Ivoire	11.4	1.4	0.7	1.2	14.7
Kenya	0.2	1.0	19.1	1.1	21.3
Morocco	7.6	2.5	15.4	5.4	31.0
Senegal	0.8	3.5	5.7	6.2	16.1
South Africa	27.2	25.9	1.1	0.6	54.9
Zimbabwe	2.2	2.5	21.6	0.5	26.8
Bangladesh	6.9	0.0	0.6	2.3	9.8
India	3.8	1.3	0.2	9.9	15.2
Indonesia	2.7	0.3	0.0	56.4	59.5
Pakistan	17.7	0.0	4.0	3.6	25.3
Philippines	2.0	10.5	1.3	1.1	15.0
Sri Lanka	2.1	16.1	9.2	0.1	27.5
Hong Kong SAR	20.0	3.3	5.3	22.0	50.6
Japan	67.0	0.1	0.3	8.1	75.5
Korea	12.1	11.1	2.8	35.0	60.9
Malaysia	4.2	0.3	6.7	42.0	53.1
Singapore	12.3	1.7	0.1	39.6	53.7
Thailand	14.7	0.0	1.7	39.7	56.0
Australia	17.4	0.1	16.2	2.4	36.1
New Zealand	5.8	0.0	1.4	5.8	13.0

Note. This table shows for each country the percentage of the variance of the real GDP growth rate explained by each estimated factor, and the commonality. The variance decomposition method is detailed in Appendix A of the paper.

Table B5: Variance decomposition with the factors estimated by PCA quartimin (international business cycles).

Country	F1 (Europe)		F2 (Latin Am.)		F3 (Northern Am.)		F4 (Dev. Asia)		Common.
	Unadj.	Adj.	Unadj.	Adj.	Unadj.	Adj.	Unadj.	Adj.	
Canada	43.2	20.0	2.1	0.2	49.6	26.2	0.0	1.2	70.7
Mexico	25.3	13.5	21.8	15.8	9.9	2.0	1.3	0.0	43.5
USA	29.3	10.1	2.2	5.8	63.0	43.1	0.2	0.2	77.5
Argentina	2.6	0.0	16.8	14.3	9.2	6.2	2.8	1.4	25.7
Bolivia	1.7	6.8	14.6	16.0	9.1	12.9	3.1	3.4	33.9
Brazil	30.4	18.3	26.8	19.1	3.3	0.1	12.4	6.2	56.7
Chile	0.5	2.0	4.7	3.4	22.5	21.7	5.4	4.1	31.2
Colombia	12.4	2.1	31.0	24.0	10.6	3.8	20.1	13.5	56.6
Costa Rica	12.2	0.3	4.9	2.2	60.1	47.5	4.4	1.7	65.0
Dominican Republic	3.7	0.6	7.8	6.3	7.1	4.2	0.1	0.0	14.5
Ecuador	2.8	0.4	33.4	30.5	0.4	0.0	5.3	3.3	37.6
El Salvador	10.4	1.0	1.4	0.3	44.7	35.2	0.1	0.1	46.2
Guatemala	21.8	5.2	22.3	15.7	32.5	17.8	3.6	0.8	57.8
Honduras	2.7	1.2	3.5	1.9	43.2	39.5	8.1	5.7	51.5
Jamaica	16.9	14.1	0.6	2.0	1.2	0.1	7.6	4.8	23.6
Panama	4.1	3.4	42.1	40.0	1.6	5.1	0.4	0.0	48.6
Paraguay	0.4	0.3	19.4	17.6	0.9	0.3	9.5	7.8	27.6
Peru	0.1	0.3	45.4	46.4	0.9	0.5	2.0	3.3	49.5
Trinidad and Tobago	9.3	7.2	24.2	21.8	0.8	0.1	2.7	5.7	35.8
Uruguay	2.2	8.2	31.5	33.1	2.4	4.0	0.5	0.4	41.0
Venezuela	12.1	5.2	19.7	15.1	2.9	0.1	9.1	5.3	33.7
Austria	65.6	51.0	1.9	0.0	9.6	0.1	7.6	2.1	67.9
Belgium	80.5	61.9	6.8	1.9	12.7	0.2	4.9	0.5	83.3
Denmark	53.7	42.7	0.0	1.1	15.4	2.5	0.0	1.1	58.3
Finland	57.1	52.7	1.9	0.2	5.5	0.0	0.2	2.8	60.1
France	85.5	66.6	2.5	0.1	16.0	0.8	3.6	0.2	86.5
Germany	59.2	39.1	1.3	0.0	18.9	3.2	9.5	3.4	66.0
Greece	53.8	45.9	0.7	0.0	7.2	0.0	0.8	0.1	53.9
Iceland	15.3	5.6	11.5	8.4	23.0	13.3	1.9	5.1	41.7
Ireland	6.2	4.5	2.7	3.7	10.9	7.5	8.1	10.6	27.9
Italy	75.2	62.2	3.3	0.3	5.2	0.6	12.7	4.9	81.0
Luxembourg	23.2	14.8	4.1	7.7	17.7	7.9	0.6	0.0	38.3
Netherlands	70.4	49.5	0.3	0.4	27.6	6.6	0.9	0.2	77.4
Norway	24.3	12.4	5.6	2.6	14.8	5.0	2.4	0.4	32.8
Portugal	58.6	44.8	0.0	1.2	9.7	0.3	12.2	5.5	65.4
Spain	68.6	60.5	0.2	0.3	10.5	0.2	0.0	2.0	71.2
Sweden	53.7	51.4	2.6	0.6	6.3	0.0	4.9	11.9	66.0
Switzerland	49.3	42.3	5.4	2.1	4.9	0.0	0.2	0.5	51.8
United Kingdom	25.6	13.9	7.9	13.2	31.8	18.3	0.3	0.0	56.2
Cameroon	0.0	0.0	2.6	3.0	0.0	0.0	3.6	3.9	6.7
Côte d'Ivoire	11.7	9.1	2.0	1.0	3.1	0.4	0.5	1.7	14.7
Kenya	0.0	2.5	1.0	0.9	16.5	19.2	1.0	1.3	21.3
Morocco	6.2	14.8	2.7	2.5	9.0	16.7	4.1	5.9	31.0
Senegal	1.5	0.0	4.2	3.0	7.2	5.3	7.0	5.4	16.1
South Africa	30.1	19.6	29.8	23.1	7.0	0.5	0.0	1.4	54.9
Zimbabwe	2.7	0.1	2.0	3.1	23.0	20.9	0.2	0.6	26.8
Bangladesh	7.0	6.6	0.1	0.0	0.0	0.8	3.1	1.9	9.8
India	4.8	1.9	2.0	0.9	1.2	0.1	11.4	9.1	15.2
Indonesia	1.6	5.4	0.5	0.3	0.1	0.0	53.2	57.1	59.5
Pakistan	17.4	19.2	0.1	0.1	0.4	5.1	5.3	3.0	25.3
Philippines	2.9	0.5	11.6	9.8	2.8	1.1	1.7	0.7	15.0
Sri Lanka	2.2	3.7	16.6	15.9	5.5	10.0	0.2	0.0	27.5
Hong Kong SAR	24.0	9.2	5.6	2.2	13.6	4.0	27.3	19.1	50.6
Japan	68.4	58.6	1.2	0.0	3.5	1.1	13.9	6.1	75.5
Korea	13.4	6.2	7.8	13.2	7.1	2.1	38.5	33.9	60.9
Malaysia	6.3	0.3	1.0	0.1	10.9	5.9	44.9	39.7	53.1
Singapore	15.1	6.5	3.3	1.0	2.4	0.0	44.5	36.9	53.7
Thailand	16.3	11.3	0.4	0.0	0.0	2.4	44.4	37.7	56.0
Australia	18.8	9.1	0.3	0.0	25.0	14.4	1.0	3.3	36.1
New Zealand	5.5	4.8	0.1	0.0	3.0	1.1	4.5	6.5	13.0

Note. This table shows for each country the percentage of the variance of the real GDP growth rate explained by each estimated factor, and the commonality. Since the factors estimated by PCA quartimin are correlated, we also report an adjusted measure of the percentage of the variance explained by each estimated factor, controlling for the influence of the other estimated factors. The variance decomposition method is detailed in Appendix A of the paper.

C Supplementary material for the application to the US economy

C.1 Data

We use the 2018:9 vintage of FRED-MD. The composition of FRED-MD is subject to minor changes over time. The 128 variables of this vintage are listed below. Most of them are retrieved from the Federal Reserve Economic Database (FRED), so their mnemonic is the same as in FRED. Some series require adjustments to the raw data available in FRED (see McCracken and Ng, 2016). The transformation code (TC) indicates how the series was transformed to ensure stationarity: (1) no transformation, (2) Δx_t, (3) $\Delta^2 x_t$, (4) $\log(x_t)$, (5) $\Delta \log(x_t)$, (6) $\Delta^2 \log(x_t)$, (7) $\Delta(x_t/x_{t-1} - 1)$. In order to work with a balanced panel, we drop 5 of 128 variables. The binary entry BP indicates whether that variable is included in the balanced panel 1960:1-2018:4.

Table C1: Output and income.

Mnemonic	Description	TC	BP
RPI	Real Personal Income	5	1
W875RX1	Real Personal Income Excluding Transfer Receipts	5	1
INDPRO	IP Index	5	1
IPFPNSS	IP: Final Products and Nonindustrial Supplies	5	1
IPFINAL	IP: Final Products (Market Group)	5	1
IPCONGD	IP: Consumer Goods	5	1
IPDCONGD	IP: Durable Consumer Goods	5	1
IPNCONGD	IP: Nondurable Consumer Goods	5	1
IPBUSEQ	IP: Business Equipment	5	1
IPMAT	IP: Materials	5	1
IPDMAT	IP: Durable Materials	5	1
IPNMAT	IP: Nondurable Materials	5	1
IPMANSICS	IP: Manufacturing (Standard Industrial Classification)	5	1
IPB51222s	IP: Residential Utilities	5	1
IPFUELS	IP: Fuels	5	1
CUMFNS	Capacity Utilization: Manufacturing	2	1

Table C2: Labor market.

Mnemonic	Description	TC	BP
HWI	Help-Wanted Index for United States	2	1
HWIURATIO	Ratio of Help Wanted to Number of Unemployed	2	1
CLF16OV	Civilian Labor Force	5	1
CE16OV	Civilian Employment	5	1
UNRATE	Civilian Unemployment Rate	2	1
UEMPMEAN	Average Duration of Unemployment (Weeks)	2	1
UEMPLT5	Civilians Unemployed - Less Than 5 Weeks	5	1
UEMP5TO14	Civilians Unemployed for 5-14 Weeks	5	1
UEMP15OV	Civilians Unemployed - 15 Weeks and Over	5	1
UEMP15T26	Civilians Unemployed for 15-26 Weeks	5	1
UEMP27OV	Civilians Unemployed for 27 Weeks and Over	5	1
CLAIMSx	Initial Claims	5	1
PAYEMS	All Employees: Total Nonfarm	5	1
USGOOD	All Employees: Goods-Producing Industries	5	1
CES1021000001	All Employees: Mining and Logging: Mining	5	1
USCONS	All Employees: Construction	5	1
MANEMP	All Employees: Manufacturing	5	1
DMANEMP	All Employees: Durable Goods	5	1
NDMANEMP	All Employees: Nondurable Goods	5	1
SRVPRD	All Employees: Service-Providing Industries	5	1
USTPU	All Employees: Trade, Transportation and Utilities	5	1
USWTRADE	All Employees: Wholesale Trade	5	1
USTRADE	All Employees: Retail Trade	5	1
USFIRE	All Employees: Financial Activities	5	1
USGOVT	All Employees: Government	5	1
CES0600000007	Average Weekly Hours: Goods-Producing	1	1
AWOTMAN	Average Weekly Overtime Hours: Manufacturing	2	1
AWHMAN	Average Weekly Hours: Manufacturing	1	1
CES0600000008	Average Hourly Earnings: Goods-Producing	6	1
CES2000000008	Average Hourly Earnings: Construction	6	1
CES3000000008	Average Hourly Earnings: Manufacturing	6	1

Table C3: Housing.

Mnemonic	Description	TC	BP
HOUST	Housing Starts: Total New Privately Owned	4	1
HOUSTNE	Housing Starts, Northeast	4	1
HOUSTMW	Housing Starts, Midwest	4	1
HOUSTS	Housing Starts, South	4	1
HOUSTW	Housing Starts, West	4	1
PERMIT	New Private Housing Permits (SAAR)	4	1
PERMITNE	New Private Housing Permits, Northeast (SAAR)	4	1
PERMITMW	New Private Housing Permits, Midwest (SAAR)	4	1
PERMITS	New Private Housing Permits, South (SAAR)	4	1
PERMITW	New Private Housing Permits, West (SAAR)	4	1

Table C4: Consumption, orders, and inventories.

Mnemonic	Description	TC	BP
DPCERA3M086SBEA	Real Personal Consumption Expenditures	5	1
CMRMTSPLx	Real Manufacturing and Trade Industries Sales	5	1
RETAILx	Retail and Food Services Sales	5	1
ACOGNO	New Orders for Consumer Goods	5	0
AMDMNOx	New Orders for Durable Goods	5	1
ANDENOx	New Orders for Nondefense Capital Good	5	0
AMDMUOx	Unfilled Orders for Durable Goods	5	1
BUSINVx	Total Business Inventories	5	1
ISRATIOx	Total Business: Inventories to Sales Ratio	2	1
UMCSENTx	Consumer Sentiment Index	2	0

Table C5: Money and credit.

Mnemonic	Description	TC	BP
M1SL	M1 Money Stock	6	1
M2SL	M2 Money Stock	6	1
M2REAL	Real M2 Money Stock	5	1
AMBSL	St. Louis Adjusted Monetary Base	6	1
TOTRESNS	Total Reserves of Depository Institutions	6	1
NONBORRES	Reserves Of Depository Institutions	7	1
BUSLOANS	Commercial and Industrial Loans	6	1
REALLN	Real Estate Loans at All Commercial Banks	6	1
NONREVSL	Total Nonrevolving Credit	6	1
CONSPI	Nonrevolving Consumer Credit to Personal Income	2	1
MZMSL	MZM Money Stock	6	1
DTCOLNVHFNM	Consumer Motor Vehicle Loans Outstanding	6	1
DTCTHFNM	Total Consumer Loans and Leases Outstanding	6	1
INVEST	Securities in Bank Credit at All Commercial Bank	6	1

Table C6: Interest rates, spreads, and exchange rates.

Mnemonic	Description	TC	BP
FEDFUNDS	Effective Federal Funds Rate	2	1
CP3Mx	3-Month AA Financial Commercial Paper Rate	2	1
TB3MS	3-Month Treasury Bill: Secondary Market Rate	2	1
TB6MS	6-Month Treasury Bill: Secondary Market Rate	2	1
GS1	1-Year Treasury Constant Maturity Rate	2	1
GS5	5-Year Treasury Constant Maturity Rate	2	1
GS10	10-Year Treasury Constant Maturity Rate	2	1
AAA	Moody's Seasoned Aaa Corporate Bond Yield	2	1
BAA	Moody's Seasoned Baa Corporate Bond Yield	2	1
COMPAPFFx	CP3Mx - FEDFUNDS	1	1
TB3SMFFM	TB3MS - FEDFUNDS	1	1
TB6SMFFM	TB6MS - FEDFUNDS	1	1
T1YFFM	GS1 - FEDFUNDS	1	1
T5YFFM	GS5 - FEDFUNDS	1	1
T10YFFM	GS10 - FEDFUNDS	1	1
AAAFFM	AAA - FEDFUNDS	1	1
BAAFFM	BAA - FEDFUNDS	1	1
TWEXMMTH	Trade Weighted U.S. Dollar Index: Major Currencies	5	0
EXSZUSx	Switzerland / U.S. Foreign Exchange Rate	5	1
EXJPUSx	Japan / U.S. Foreign Exchange Rate	5	1
EXUSUKx	U.S. / U.K. Foreign Exchange Rate	5	1
EXCAUSx	Canada / U.S. Foreign Exchange Rate	5	1

Table C7: Prices.

Mnemonic	Description	TC	BP
WPSFD49207	PPI: Finished Goods	6	1
WPSFD49502	PPI: Finished Consumer Goods	6	1
WPSID61	PPI: Intermediate Materials	6	1
WPSID62	PPI: Crude Materials	6	1
OILPRICEx	Crude Oil, Spliced WTI and Cushing	6	1
PPICMM	PPI: Metals and Metal Products	6	1
CPIAUCSL	CPI: All Items	6	1
CPIAPPSL	CPI: Apparel	6	1
CPITRNSL	CPI: Transportation	6	1
CPIMEDSL	CPI: Medical Care	6	1
CUSR0000SAC	CPI: Commodities	6	1
CUSR0000SAD	CPI: Durables	6	1
CUSR0000SAS	CPI: Services	6	1
CPIULFSL	CPI: All Items Less Food	6	1
CUSR0000SA0L2	CPI: All Items Less Shelter	6	1
CUSR0000SA0L5	CPI: All Items Less Medical Care	6	1
PCEPI	Personal Consumption Expenditures: Chain Index	6	1
DDURRG3M086SBEA	Personal Consumption Expenditures: Durable Goods	6	1
DNDGRG3M086SBEA	Personal Consumption Expenditures: Nondurable Goods	6	1
DSERRG3M086SBEA	Personal Consumption Expenditures: Services	6	1

Table C8: Stock market.

Mnemonic	Description	TC	BP
S&P 500	S&P's Common Stock Price Index: Composite	5	1
S&P: indust	S&P's Common Stock Price Index: Industrials	5	1
S&P div yield	S&P's Composite Common Stock: Dividend Yield	2	1
S&P PE ratio	S&P's Composite Common Stock: Price-Earnings Ratio	5	1
VXOCLSx	VXO	1	0

C.2 Additional results

Table C9: Selected number of factors (FRED-MD).

Criterion	IC_{p1}	IC_{p2}	IC_{p3}	PC_{p1}	PC_{p2}	PC_{p3}
r^*	8	8	10	9	9	10

Note. Number of factors selected by each of the six main Bai and Ng (2002) information criteria, allowing for at most ten factors.

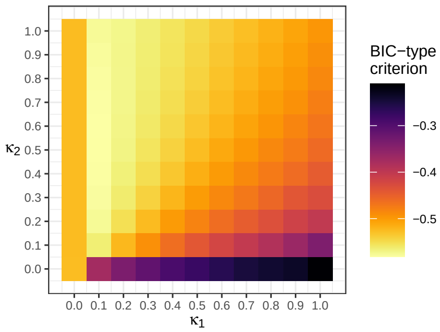


Figure C1: Grid search of the SPCA hyperparameters κ_1 and κ_2 (FRED-MD).

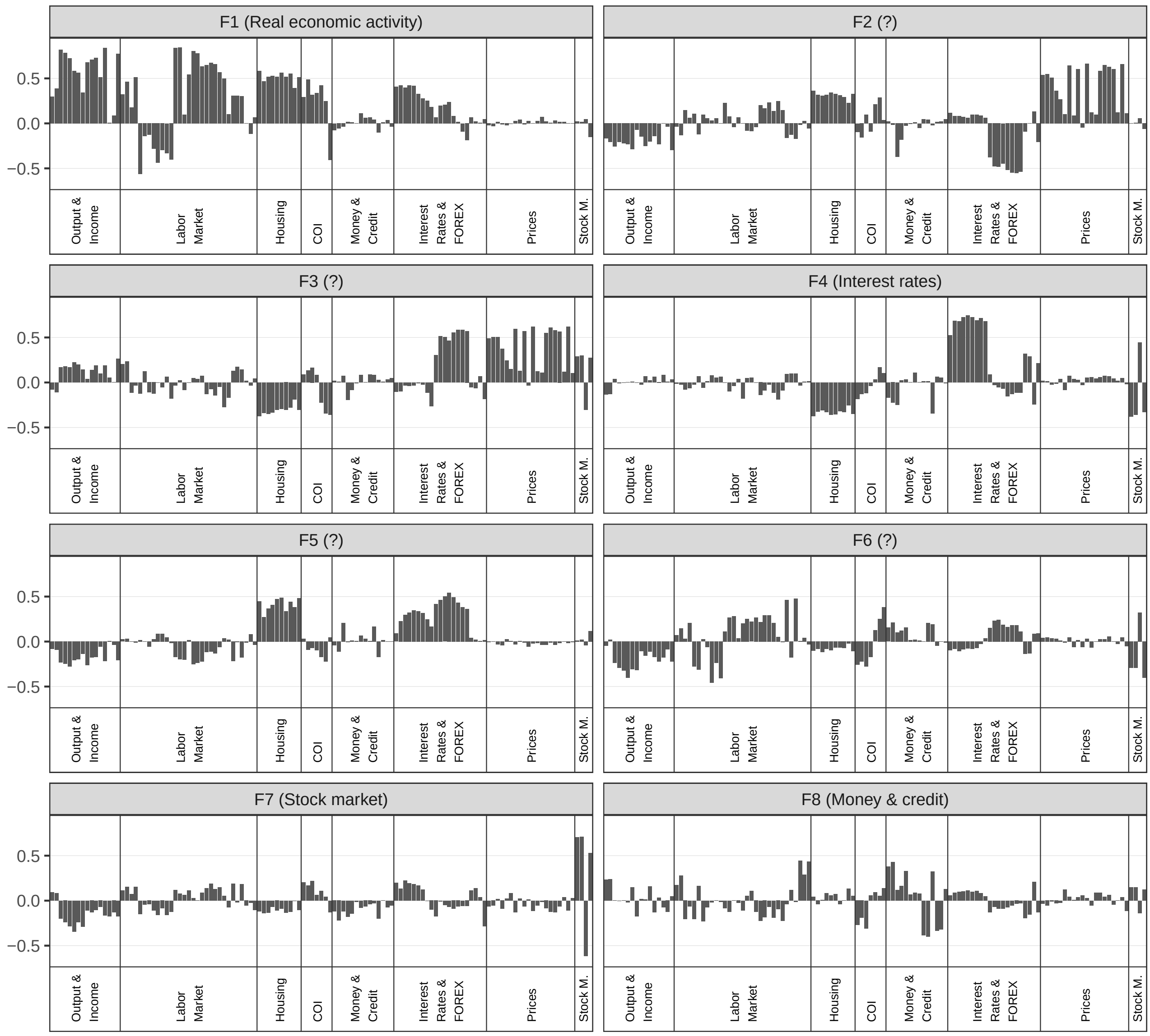


Figure C2: Factor loadings estimated with PCA (FRED-MD).

Note. The interpretable factors are labeled with their economic interpretation, and the uninterpretable factors with a question mark. Since F and Λ are identified up to a column sign change, for each factor, we impose the largest loading in absolute value to be positive.

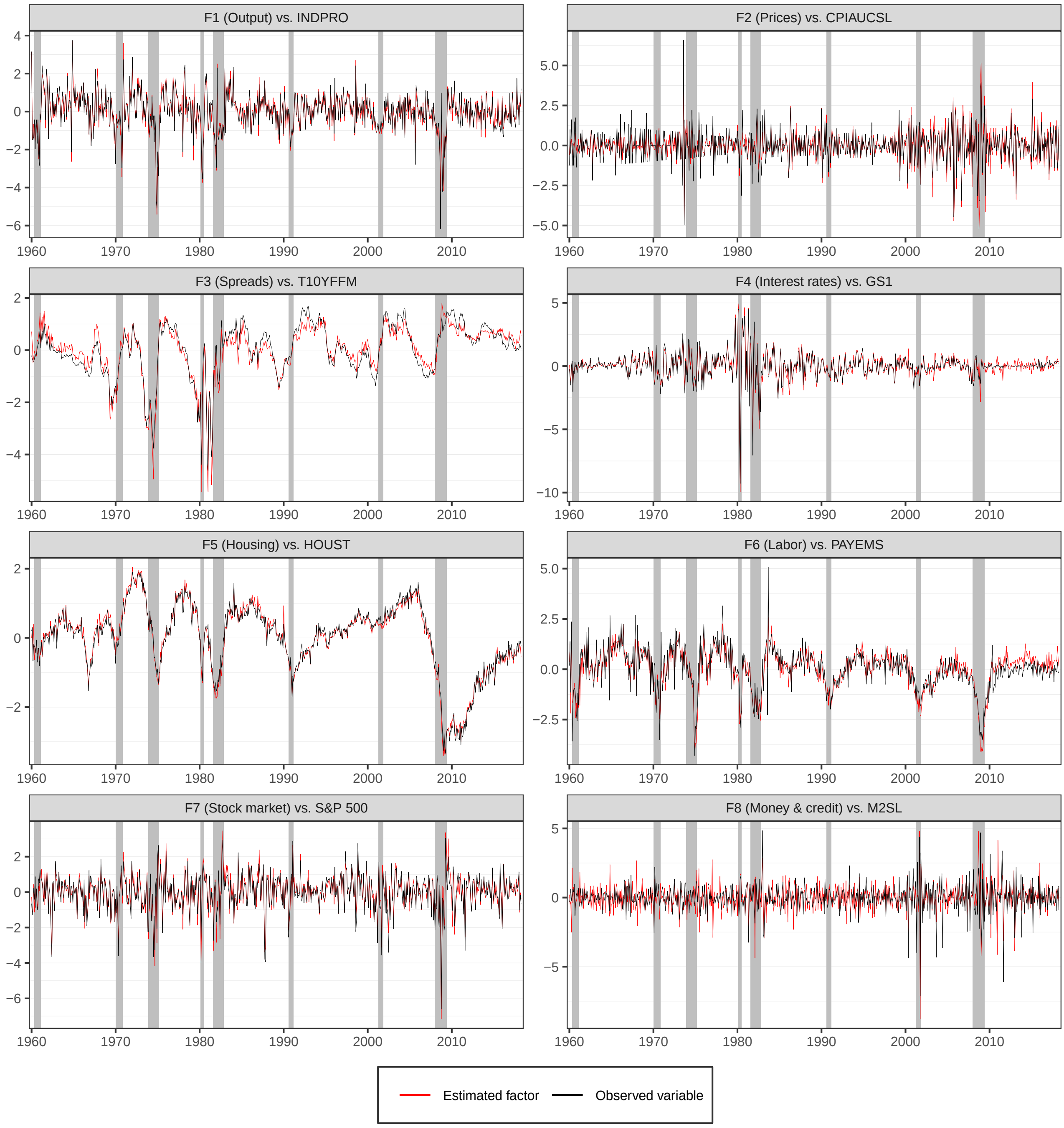


Figure C3: Factors estimated with SPCA, compared with some representative observed variables (FRED-MD).

Note. Shaded areas indicate periods of recession as defined by the National Bureau of Economic Research. The observed variables are stationarized using the transformations recommended by McCracken and Ng (2016). The correlations between the factors estimated with SPCA and the representative observed variables are respectively 0.96, 0.89, 0.95, 0.96, 0.99, 0.91, 0.94, 0.61.

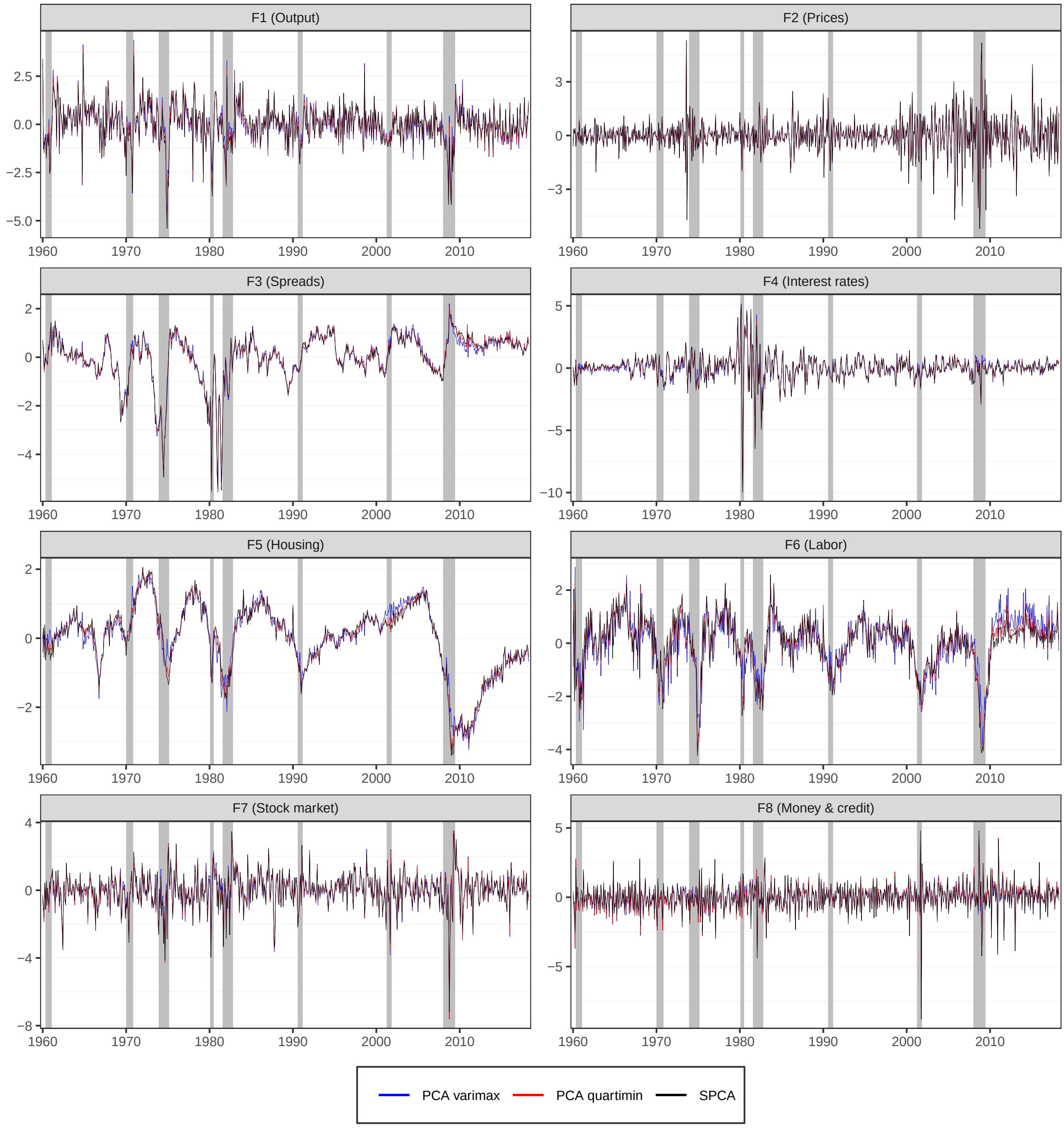


Figure C4: Factors estimated with PCA varimax, PCA quartimin, and SPCA (FRED-MD).

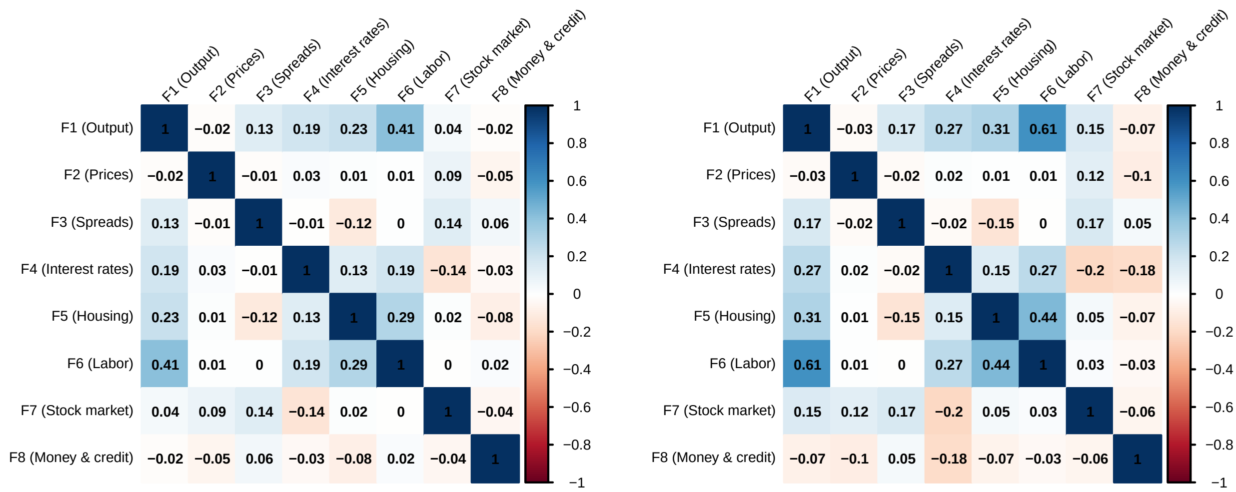
Note. Shaded areas indicate periods of recession as defined by the National Bureau of Economic Research.

Table C10: Correlations between the estimated factors (FRED-MD).

Factors	PCA varimax vs. PCA quartimin	PCA varimax vs. SPCA	PCA quartimin vs. SPCA
F1 (Output)	0.99	0.95	0.99
F2 (Prices)	1.00	1.00	1.00
F3 (Spreads)	0.99	0.99	1.00
F4 (Interest rates)	0.99	0.98	1.00
F5 (Housing)	0.99	0.97	1.00
F6 (Labor)	0.93	0.87	0.99
F7 (Stock market)	0.99	0.96	0.98
F8 (Money & credit)	0.99	0.95	0.96

Table C11: Correlations between the estimated factors and some representative observed variables (FRED-MD).

Pairs	PCA varimax	PCA quartimin	SPCA
F1 (Output) vs. INDPRO	0.92	0.95	0.96
F2 (Prices) vs. CPIAUCSL	0.89	0.89	0.89
F3 (Spreads) vs. T10YFFM	0.93	0.94	0.95
F4 (Interest rates) vs. GS1	0.93	0.95	0.96
F5 (Housing) vs. HOUST	0.97	0.99	0.99
F6 (Labor) vs. PAYEMS	0.72	0.87	0.91
F7 (Stock market) vs. S&P 500	0.90	0.91	0.94
F8 (Money & credit) vs. M2SL	0.51	0.51	0.61



(a) PCA quartimin (b) SPCA

Figure C5: Correlation matrices of the estimated factors (FRED-MD).

Note. The correlation matrices of the factors estimated with PCA and PCA varimax are not displayed since they are simply the identity matrix, by construction.

References

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