A Hidden Markov Model Approach to Information-Based Trading:

Theory and Applications

Online Supplementary Appendix

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Online Supplementary Appendix

S.1. Proof of Likelihood Function (2)

Mathematically, this HMM is described by

$$\Pr\left(H_t \middle| H_{(t-1)}\right) = \Pr(H_t|H_{t-1}) \quad \text{and} \quad \Pr\left(X_t \middle| X_{(t-1)}, H_{(t)}\right) = \Pr(X_t|H_t).$$

Since $\Pr\left(X_{(t+1)}, H_{(t+1)}\right) = \Pr(X_{t+1}|H_{t+1})\Pr(H_{t+1}|H_t)\Pr\left(X_{(t)}, H_{(t)}\right)$, the joint distribution of $X_{(t)}$ and $H_{(t)}$ can be written as

$$\Pr\left(X_{(t)}, H_{(t)}\right) = \Pr(H_1)\prod_{k=2}^{t}\Pr(H_k|H_{k-1})\prod_{k=1}^{t}\Pr(X_k|H_k).$$

Summing over $H_{(t-1)}$ gives us

$$\Pr\left(X_{(t+1)}, H_t, H_{t+1}\right) = \Pr\left(X_{(t)}, H_t\right)\Pr(H_{t+1}|H_t)\Pr(X_{t+1}|H_{t+1}).$$

We refer η_{t+1} as the vector of forward probabilities, whose $\left((i-1)n+j\right)$-th element is

$$\begin{aligned}
\eta_{i,j;t+1} &= \Pr\left(X_{(t+1)} = x_{(t+1)}, H_{b;t+1} = i, H_{s;t+1} = j\right) \\
&= \sum_{k=1}^{m}\sum_{l=1}^{n}\Pr\left(X_{(t+1)} = x_{(t+1)}, H_{b;t} = k, H_{s;t} = l, H_{b;t+1} = i, H_{s;t+1} = j\right) \\
&= \sum_{k=1}^{m}\sum_{l=1}^{n}\Pr\left(X_{(t)} = x_{(t)}, H_{b;t} = k, H_{s;t} = l\right)\Pr\left(H_{b;t+1} = i, H_{s;t+1} = j \middle| H_{b;t} = k, H_{s;t} = l\right) \\
&\qquad \times \Pr\left(X_{t+1} = x_{t+1} \middle| H_{b;t+1} = i, H_{s;t+1} = j\right) \\
&= \sum_{k=1}^{m}\sum_{l=1}^{n}\eta_{k,l;t}\,\gamma_{i,j;k,l}\,p_{i,j}(x_{t+1}).
\end{aligned}$$

It can be rewritten in a matrix in the form of $\eta_{t+1} = \eta_t\Gamma P(x_{t+1})$. In addition, $\eta_1 = u_1 P(x_1)$. Therefore, we can compute the likelihood function in (2) recursively in terms of the forward probabilities as follows

$$L\left(\boldsymbol{\Theta}\middle|x_{(T)}\right) = \eta_T\mathbf{1} = \eta_{T-1}\Gamma P(x_T)\mathbf{1} = u_1 P(x_1)\Gamma P(x_2)\Gamma P(x_3)\cdots\Gamma P(x_T)\mathbf{1}.$$

S.2. Estimation of the HMM by the Expectation and Maximization Algorithm

We apply the Baum-Welch algorithm (see Baum *et al.,* 1970) to estimate the HMM. It is an iterative method for maximum likelihood estimation when there are missing data. It exploits the fact that the Complete-Data Log-Likelihood (CDLL) can be directly applied to maximization even if the likelihood of the observed data cannot be applied. In our case, we regard the hidden states as missing data while the CDLL is the log-likelihood of parameter set $\mathbf{\Theta}$ based on observed time series of buy and sell order flows and the unobservable time series of states, i.e. $\log(\Pr(X_{(T)} = x_{(T)}, H_{(T)} = h_{(T)} | \mathbf{\Theta}))$, where $h_{(T)}$ is a time series realization of state variable H_t with t ranging from 1 to T. Denote η_{t+1} as the vector of forward probabilities, whose $\big((i-1)n+j\big)$-th element is

$$\eta_{i,j;t+1} = \Pr\big(X_{(t+1)} = x_{(t+1)}, H_{b;t+1} = i, H_{s;t+1} = j\big).$$

We have $\eta_1 = u_1 P(x_1)$ and $\eta_t = \eta_{t-1}\Gamma P(x_t)$ for $t = 2, 3, \dots, T$. To apply the Expectation and Maximization (EM) algorithm, we also define $\zeta'_t = \Gamma P(x_{t+1})\zeta'_{t+1}$ as the vector of backward probabilities for $t = 1, 2, \dots, T-1$ with $\zeta_T = \mathbf{1}'$, where the $((i-1)n+j)$-th element of ζ_t is

$$\zeta_{i,j;t} = \Pr(X_{t+1} = x_{t+1}, X_{t+2} = x_{t+2}, \dots, X_T = x_T | H_{b;t} = i, H_{s;t} = j).$$

Further, let $z_{i,j;t}$ and $z_{i,j;k,l;t}$ be zero-one variables that

$$z_{i,j;t} = 1 \text{ if and only if } H_{b;t} = i, H_{s;t} = j,$$
$$z_{i,j;k,l;t} = 1 \text{ if and only if } H_{b;t-1} = i, H_{s;t-1} = j, H_{b;t} = k, H_{s;t} = l.$$

With this notation, the CDLL of the HMM is given by

$$\log\big(\Pr\big(x_{(T)}, h_{(T)}\big)\big) = \sum_{i=1}^{m}\sum_{j=1}^{n} z_{i,j;1}\log(u_{i,j;1}) + \sum_{t=2}^{T}\sum_{i,k=1}^{m}\sum_{j,l=1}^{n} z_{i,j;k,l;t}\log(\gamma_{i,j;k,l}) + \sum_{t=1}^{T}\sum_{i=1}^{m}\sum_{j=1}^{n} z_{i,j;t}\log(p_{i,j}(x_t)).$$

We use the EM algorithm to estimate the HMM as follows:[1]

- E Step: Compute the conditional expectations of the missing data, given the observations $x_{(T)}$ and the current estimate of $\mathbf{\Theta}$. Specifically, conditional expectations of $z_{i,j;t}$ and $z_{i,j;k,l;t}$ are estimated

[1] For details of the algorithm, see Cappé *et al.* (2005), and Zucchini and MacDonald (2009).

by:

$$\bar{z}_{i,j;t} = \Pr\left(H_{b;t} = i, H_{s;t} = j \middle| x_{(T)}, \mathbf{\Theta}\right) = \frac{\eta_{i,j;t}\zeta_{i.j;t}}{L(\mathbf{\Theta}|x_{(T)})},$$

$$\bar{z}_{i,j;k,l;t} = \Pr\left(H_{b;t-1} = i, H_{s;t-1} = j, H_{b;t} = k, H_{s;t} = l | x_{(T)}, \mathbf{\Theta}\right) = \frac{\eta_{i,j;t-1}\gamma_{i,j;k,l}p_{k,l}(x_t)\zeta_{k,l;t}}{L(\mathbf{\Theta}|x_{(T)})}.$$

- M Step: Maximize the CDLL, where the missing data are replaced by their conditional expectations, to determine the estimate of $\mathbf{\Theta}$. Thus, we replace all $z_{i,j;t}$ and $z_{i,j;k,l;t}$ in CDLL by their conditional means $\bar{z}_{i,j;t}$ and $\bar{z}_{i,j;k,l;t}$, and maximize it with respect to u_1, Γ, and $\lambda_{b;i}$ and $\lambda_{s;j}$. The solution to the maximization problem consists of

$$u_{i,j;1} = \bar{z}_{i,j;1}, \qquad \gamma_{i,j;k,l} = \frac{\sum_{t=2}^{T} \bar{z}_{i,j;k,l;t}}{\sum_{k'=1}^{m}\sum_{l'=1}^{n}\sum_{t=2}^{T} \bar{z}_{i,j;k',l';t}},$$

$$\lambda_{b,i} = \frac{\sum_{j=1}^{n}\sum_{t=1}^{T} \bar{z}_{i,j;t}\, b_t}{\sum_{j=1}^{n}\sum_{t=1}^{T} \bar{z}_{i,j;t}} \quad \text{and} \quad \lambda_{s,j} = \frac{\sum_{i=1}^{m}\sum_{t=1}^{T} \bar{z}_{i,j;t}\, s_t}{\sum_{i=1}^{m}\sum_{t=1}^{T} \bar{z}_{i,j;t}}.$$

The above E and M steps are repeated many times until some convergence criterion has been satisfied, for instance the improvement in the CDLL is less than 10^{-6}. This EM algorithm provides us with three sets of parameter estimates: u_1, Γ, and $\lambda_{b;i}$ and $\lambda_{s;j}$. Once u_1 and Γ are estimated, we have $u_t = u_1\Gamma^{t-1}$.

Applying Bayes' rule, the posterior distribution of states H_t in (1) can be calculated by

$$\Pr\left(H_{b;t} = i, H_{s;t} = j \middle| X_{(T)} = x_{(T)}\right) = \frac{\Pr\left(X_{(T)} = x_{(T)}, H_{1;t} = i, H_{2;t} = j\right)}{\Pr\left(X_{(T)} = x_{(T)}\right)} = \frac{\eta_{i,j;t}\zeta_{i,j;t}}{L(\mathbf{\Theta}|x_{(T)})} = \bar{z}_{i,j;t}. \tag{A1}$$

When implementing the EM algorithm for estimating the HMM, it is relatively convenient to find plausible starting values for the initial distribution of states and the transition matrix. One strategy is to assign a uniform starting value to all the elements of the initial state distribution and the transition matrix. If the number of states is N, we assign $u_1 = \frac{1}{N}\mathbf{1'}$ and $\Gamma = \frac{1}{N}\mathbb{N}$, where $\mathbb{N}$ is the matrix of size $N \times N$ with all elements equal to 1. In order to improve the convergence speed, we run an N-means clustering on the observed buys and sells and then use the centers of the clusters as the initial starting values of the state-dependent order arrival rates.

S.3. K-means Clustering Analysis and the Jump Method in Finding the Number of Clusters

In modeling information-based trading, the concept of clusters is mentioned in Duarte and Young (2009) in describing the sample distribution of order flows. With evidence from the T-bill market, Akay *et al.* (2012) conclude that *PIN* identifies trading clusters and clustering depends on the market conditions. Trading activities of a particular day depend on the information environment of the market. Therefore, trading days with common features are very likely to have a similar information environment. Clustering analysis is an iterative process of knowledge discovery that group a set of objects in such a way that objects in the same group (called cluster) are more similar to each other than to those in other groups (clusters). We use *k*-means clustering (see MacQueen, 1967) as it is one of the simplest and widely-adopted algorithms that solve clustering problems. Given a set of observations $y = (y_1, y_2, \cdots, y_T)'$, where each observation is a *d*-dimensional vector, *k*-means clustering partitions the *T* observations into *k* groups $G = \{G_1, G_2, \cdots, G_k\}$ to minimize the within-cluster sum of squares

$$argmin_G \sum_{i=1}^{k} \sum_{y_j \in G_i} ||y_j - c_i||^2,$$

where c_i is the mean of points in G_i, called the center of cluster *i.* Based on the centers of the *k* clusters, any out-of-sample observation is assigned to the cluster with the shortest distance to the center. In the case of our application, the observations are the daily numbers of trading imbalances or balanced trades, which are partitioned into *k* clusters. Each cluster represents a regime of trading activities and therefore is associated with certain information characteristics. For each hidden state in the HMM, its associated expected number of trading imbalances or balanced trades is treated as an out-of-sample observation, which can be assigned to the cluster with the nearest center. Consequently, each hidden state is classified into a unique cluster.

The difficulty in cluster analysis is identifying the number of clusters *k*. Sugar and James (2003) develop a simple, yet powerful nonparametric method for choosing the number of clusters based on a so-called jump method with a rigorous theoretical justification.[2] They demonstrate its effectiveness not only for choosing the number of clusters but also for identifying the underlying structure on a wide range of simulated and real world datasets. We therefore adopt this jump method for clustering and choosing the number of clusters in our application.

S.4. State Prediction, Forecast Distribution of Order Flows and Forecast Pseudo-Residual

In the HMM approach, it is feasible to perform state prediction by deriving the conditional distribution of states H_t for $t > T$ as follows

$$\Pr\left(H_{b;T+h} = i, H_{s;T+h} = j \middle| x_{(T)}\right)$$

$$= \frac{\sum_{k=1}^{m}\sum_{l=1}^{n} \Pr\left(H_{b;T+h} = i, H_{s;T+h} = j \middle| H_{b;T} = k, H_{s;T} = l\right) \Pr\left(H_{b;T} = k, H_{s;T} = l, x_{(T)}\right)}{L\left(\mathbf{\Theta} \middle| x_{(T)}\right)},$$

where $\Pr\left(H_{b;T+h} = i, H_{s;T+h} = j \middle| H_{b;T} = k, H_{s;T} = l\right)$ is an element of the h-step transition matrix Γ^h, $\Pr\left(H_{b;T} = k, H_{s;T} = l, x_{(T)}\right)$ is the $\left((k-1)n + l\right)$-th element of the forward probability η_T defined in Appendix S.1, and $L\left(\mathbf{\Theta} \middle| x_{(T)}\right)$ the likelihood function of the HMM defined in (2). The forecast conditional distribution of the order flows with forecast horizon of h can be derived through

$$\Pr\left(X_{T+h} = x_{T+h} \middle| x_{(T)}\right) = \frac{\sum_{k=1}^{m}\sum_{l=1}^{n} \Pr\left(H_{b;T} = k, H_{s;T} = l, x_{(T)}\right) \Gamma^h P(x_{T+h}) \mathbf{1}}{L\left(\mathbf{\Theta} \middle| x_{(T)}\right)}.$$

Based on this forecasted conditional distribution, we can appropriately define the forecast pseudo-residual to perform the out-of-sample analysis as illustrated in Dunn and Smyth (1996) and Zucchini and MacDonald (2009). Let Φ be the cumulative distribution function of the standard normal distribution and Y a random variable with cumulative distribution function F, then $Z \equiv \Phi^{-1}(F(Y))$ is a standard normal variable. In our HMM, the random variable X_{T+h} is discrete and the forecast normal pseudo-residual segment is defined as

[2] We thank Gareth M. James for making the R code of implementing the jump method available on his personal website.

$$[z_{T+h}^{-}; z_{T+h}] = \left[\Phi^{-1}\left(F_{X_{T+h}}(x_{T+h}^{-})\right); \Phi^{-1}\left(F_{X_{T+h}}(x_{T+h})\right)\right]$$

$$= \left[\Phi^{-1}\left(\Pr\left(X_{T+h} < x_{T+h} \middle| X_{(T)} = x_{(T)}\right)\right); \Phi^{-1}\left(\Pr\left(X_{T+h} \le x_{T+h} \middle| X_{(T)} = x_{(T)}\right)\right)\right],$$

where x_{T+h}^{-} denotes the greatest realization that is strictly less than x_{T+h}. This forecast pseudo-residual segments can be interpreted as interval-censored realizations of a standard normal distribution, if the fitted HMM model is valid in the out-of-sample period. If the forecast pseudo-residual segment is extreme, say lying entirely within the top or bottom 0.5% of the standard normal distribution, i.e. $min(|z_{T+h}^{-}|, |z_{T+h}|) > 2.576 = \Phi^{-1}(0.995)$, the observation $X_{T+h} = x_{T+h}$ is an outlier or the candidate model no longer provides an acceptable description of the series.

S.5. Characteristics of the Hidden States

It may be interesting to know the main characteristics of the hidden states of the HMM for the 120 sample stocks. Limited by the length of the paper, we report them in this appendix.

Important quantities associated with a Markov chain are transition probabilities. The transition probabilities between hidden states in the HMM provide us with a forward-looking indicator of the information evolution. Although there are on average 26.27 states, some of them have common features and therefore can be grouped into the same aggregate state. As discussed in the last subsection, we consider four aggregate states and each of them includes only one type of states. For notation convenience we label liquidity, private information, public information, and private and public information aggregate states from 1 to 4, respectively. Then, we can introduce average transition probability from a state in aggregate state J to aggregate state K:

$$\alpha_{JK} = \frac{1}{\text{number of states in set } J} \sum_{\substack{(i,j) \in \text{aggregate state } J \\ (l,k) \in \text{aggregate state } K}} \gamma_{i,j;l,k} \text{ for } J \text{ and } K = 1, 2, 3, 4,$$

where $\gamma_{i,j;l,k}$ is an element in the original transition matrix Γ denoting the probability of state being (l, k) on day $t + 1$ conditional on it being (i, j) on day t and $\sum_{(l,k) \in \text{aggregate state } K} \gamma_{i,j;l,k}$, denotes the transition probability from state (i, j) to aggregate state K. Roughly speaking, α_{JK} reflects the probability of being aggregate state K on day $t + 1$ conditional on being aggregate state J on day t. Figure S-1 below uses boxplots to show the statistical characteristics of these average transition probabilities across the 120 sample stocks. They exhibit strong cross-sectional variations. For instance, the first boxplot in Figure S-1 (I) depicts the descriptive statistics of α_{11} of the 120 sample stocks, i.e. the statistics of 120 average transition probabilities from a liquidity state to liquidity aggregate state. The central rectangle shows the first quartile and the third quartile are 0.25 and 0.5352, respectively, while the segment inside the rectangle shows the median is 0.3839. The "whiskers" above and below the rectangle indicate the minimum and maximum of α_{11} are 0.1805 and 0.6712, respectively. We can also see that there are outliers of α_{JK} labelled by circles in the first, third and fourth boxplots in Figure S-1 (II), the second and third boxplots in Figure S-1 (III), and the first, second and fourth boxplots in Figure S-1 (IV).

In these cases, the whisker on the appropriate side is taken to be 1.5 times of interquartile range (the interval between the first quartile and the third quartile). We can see that the average transition probabilities to and from liquidity states are non-trivial in the HMM. Therefore, the HMM can accommodate jumps in liquidity states that neither static models nor AR-type dynamics can capture. As shown in the second boxplot of Figure S-1 (II), there is a significant probability for two consecutive trading days being associated with privately informed trading, i.e. $\alpha_{22} > 0$. It is consistent with the information diffusion hypothesis of Hong and Stein (1999) that private information diffuses gradually across the investing public. The HMM also identifies non-trivial transition probabilities from a private information state to a public information state (i.e. $\alpha_{23} > 0$) and from a public information state to a state with private and public information (i.e. $\alpha_{34} > 0$). It therefore empirically supports the existence of pre-announcement and event-period private information, which are theoretically introduced by Kim and Verrecchia (1997).

It is also interesting to know the stationary distribution of states, which can be calculated based on the transition matrix Γ through the relationship $\delta\Gamma = \delta$. The stationary probability of being aggregate state Kcan be calculated by:

$$\mathcal{D}_K = \frac{1}{\text{number of states in set K}} \sum_{(i,j)\, \in \text{aggregate state } K} \delta_{i,j} \text{ for } K = 1, 2, 3, 4,$$

where $\delta_{i,j}$ is the stationary probability of being state (i,j). Figure S-2 (I) below provides the boxplots of the stationary probabilities of being the four aggregate states for the 120 sample stocks. Figure S-2 (II) shows the boxplots of the sample kurtosis of buy and sell orders under the stationary state distribution. For about half of the sample stocks, buy and/or sell order flows exhibit fat tails, as evidenced by their positive excess kurtosis. Thus, even in equilibrium (stationary states) the order flows are mixed with liquidity, private information, and SOS trading. The endogenous switching across information states causes fat tails in order flows to appear.

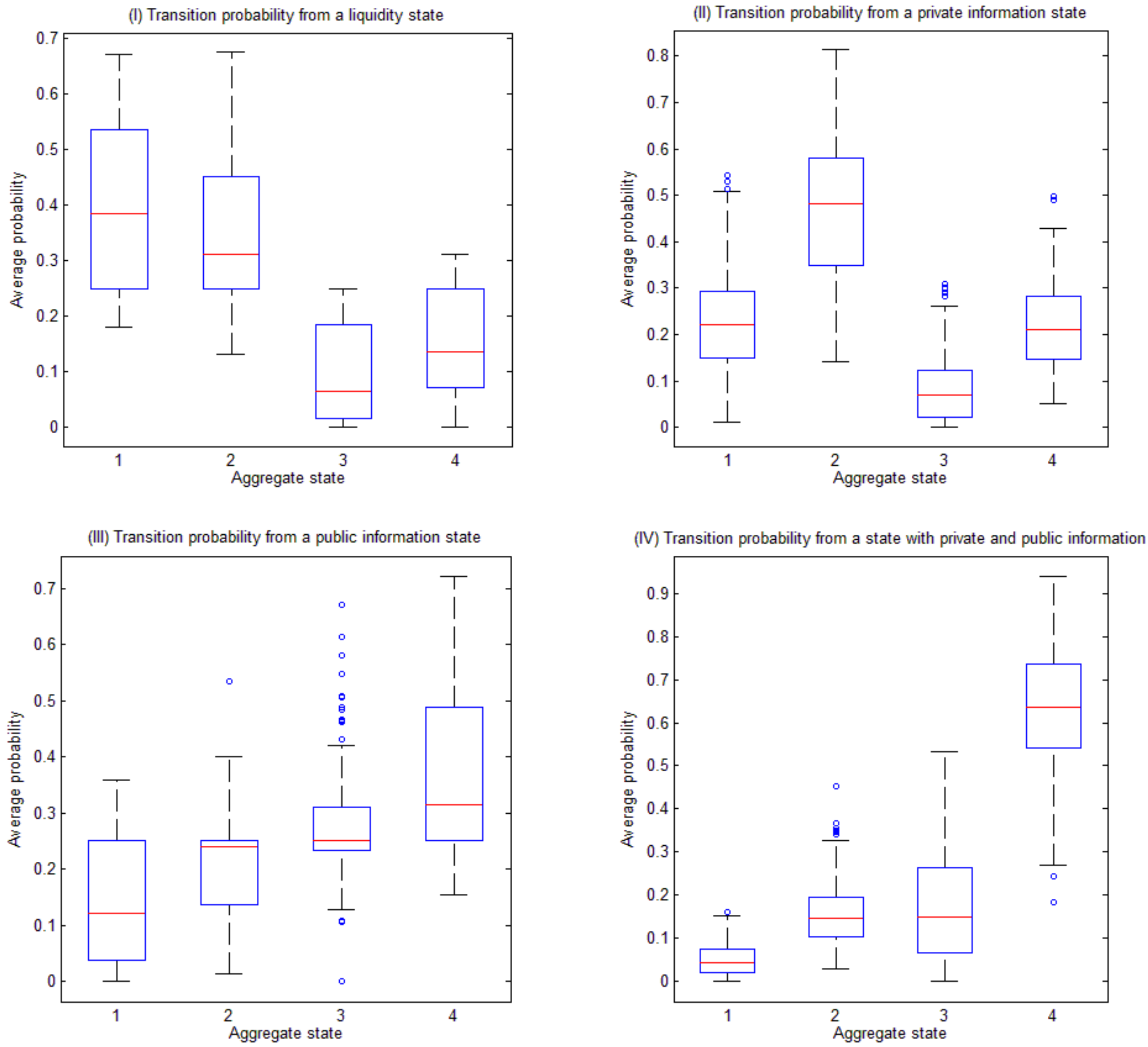

Figure S-1. Descriptive statistics of average transition probabilities. This figure shows the boxplots of the average transition probabilities to the four aggregate states for the 120 sample stocks. Liquidity, private information, public information, and private and public information aggregate states are labeled by 1, 2, 3 and 4, respectively. The average transition probabilities from a state in aggregate state J to aggregate state K is estimated by

$$\alpha_{JK} = \frac{1}{\text{number of states in set } J} \sum_{\substack{(i,j) \in \text{aggregate state } J \\ (l,k) \in \text{aggregate state } K}} \gamma_{i,j;l,k} \quad \text{for } J \text{ and } K = 1, 2, 3, 4,$$

where $\gamma_{i,j;l,k}$ is an element in the original transition matrix Γ denoting the probability of state being (l,k) on day $t+1$ conditional on it being (i,j) on day t.

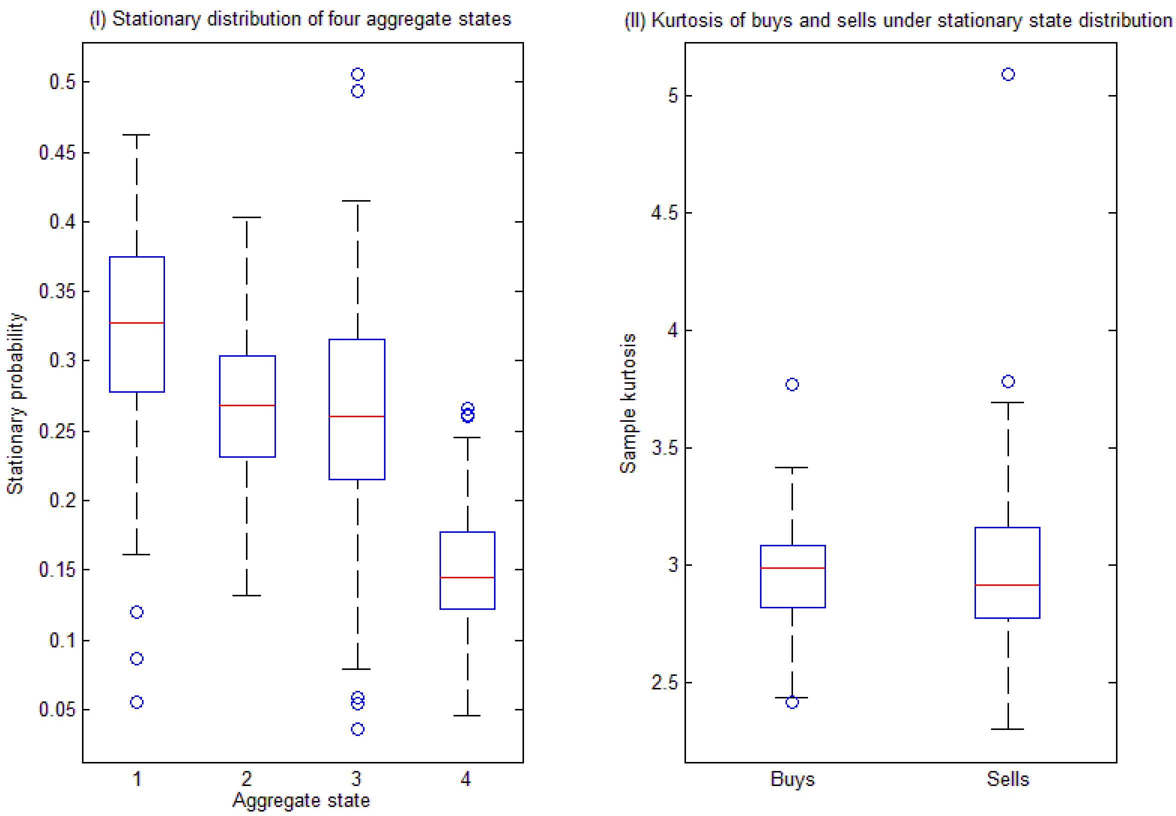

Figure S-2. Descriptive statistics of stationary state distribution and stationary kurtosis of buy and sell orders. The stationary probability of states being aggregate state K is calculated through $\frac{1}{\text{number of states in set } K}\sum_{(i,j)\,\in\,\text{aggregate state } K} \delta_{i,j}$, where $\delta_{i,j}$ is the stationary probability of being state (i,j). Figure S-2 (I) provides the boxplots of the stationary probabilities of four aggregate states, where liquidity, private information, public information, and private and public information aggregate states are labeled by 1, 2, 3 and 4, respectively. Figure S-2 (II) shows the boxplots of the sample kurtosis of buys and sells under the stationary state distribution.

S.6. Tables and Figures of Simulation Results

This section of appendix presents figures and tables discussed in Section 3.

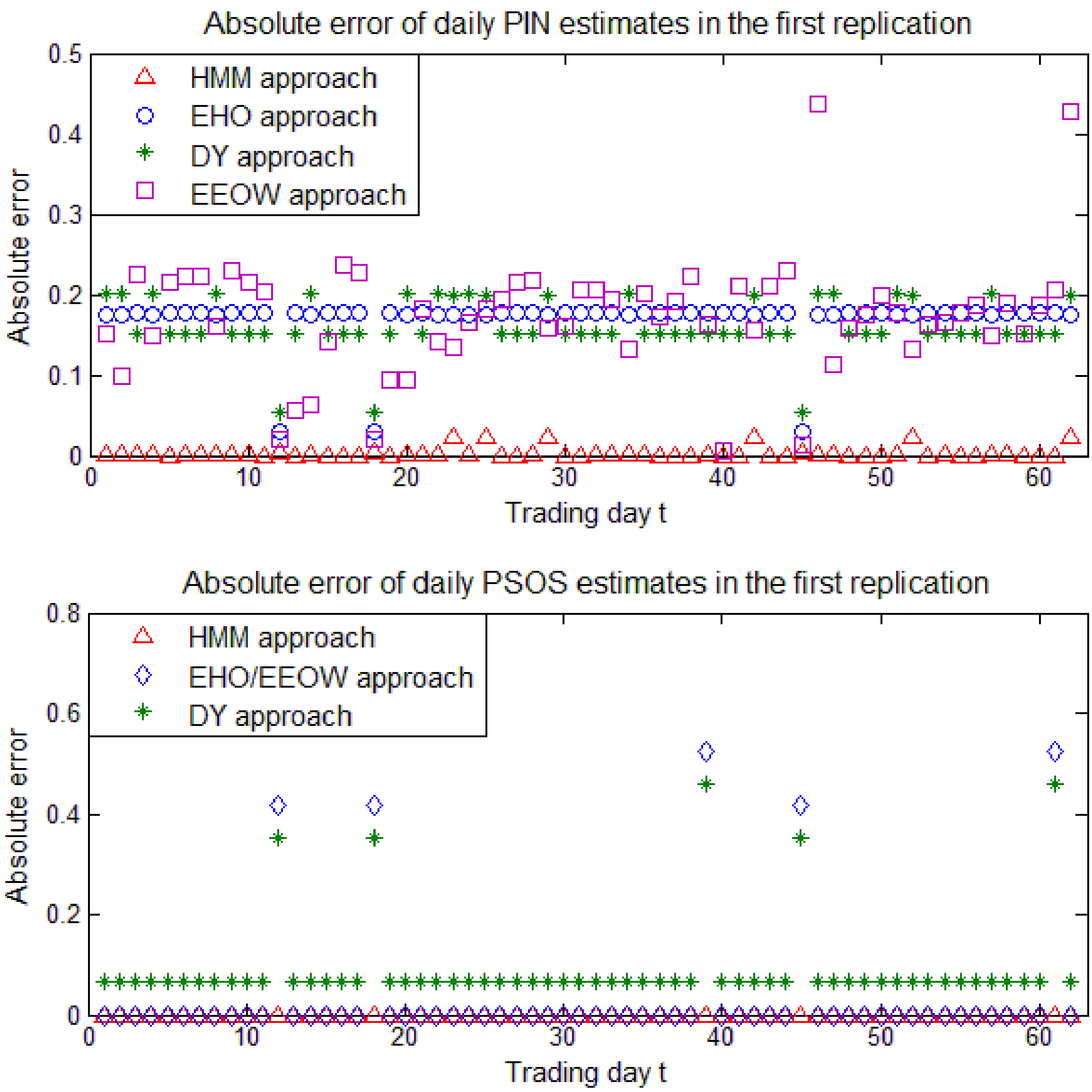

Figure S-3. Absolute error of daily *PIN* and *PSOS* estimates. The hypothetical trading data over 63 trading days are simulated according to the DY model with parameters $\alpha = 0.28$, $\delta = 0.3$, $\mu_b = 132$, $\mu_s = 133$, $\varepsilon_b = 121$, $\varepsilon_s = 123$, $\theta = \theta' = 0.1$, $\upsilon_b = 139$, $\upsilon_s = 131$ (Scenario 2.1 of Table S-I). The upper part plots the absolute errors in daily *PIN* estimates, i.e., $\left|\widehat{PIN}_t^{(1)} - PIN_t^{(1)}\right|$, and the lower part plots the absolute errors in daily *PSOS* estimates, i.e., $\left|\widehat{PSOS}_t^{(1)} - PSOS_t^{(1)}\right|$, where $PIN_t^{(1)}$ and $PSOS_t^{(1)}$ are the true daily *PIN* and *PSOS* of day *t* in the first replication, and $\widehat{PIN}_t^{(1)}$ and $\widehat{PSOS}_t^{(1)}$ denote the estimates based on the candidate estimation approach.

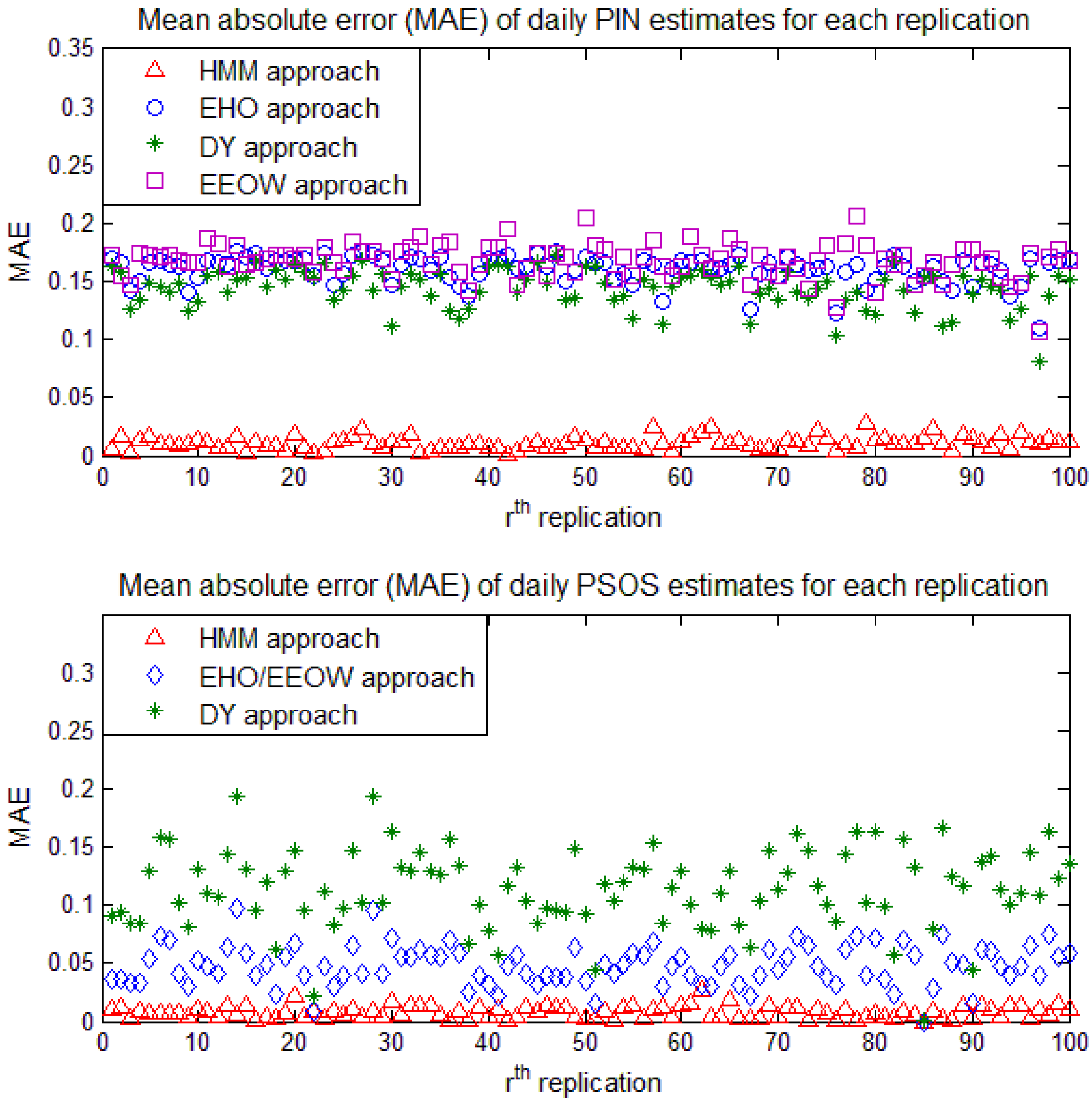

Figure S-4. Mean absolute error of the daily estimates of *PIN* and *PSOS* over 100 replications. The hypothetical trading data of 63 trading days are simulated according to the DY model with parameters $\alpha = 0.28$, $\delta = 0.3$, $\mu_b = 132$, $\mu_s = 133$, $\varepsilon_b = 121$, $\varepsilon_s = 123$, $\theta = \theta' = 0.1$, $\upsilon_b = 139$, $\upsilon_s = 131$ (Scenario 2.1 of Table S-I). In the r^{th} replication, the mean absolute error is given by $MAE^{(r)}_{PIN_t^{[0,T]}} = \frac{1}{T}\sum_{t=1}^{T}\left|\widehat{PIN}_t^{(r)} - PIN_t^{(r)}\right|$, and $MAE^{(r)}_{PSOS_t^{[0,T]}} = \frac{1}{T}\sum_{t=1}^{T}\left|\widehat{PSOS}_t^{(r)} - PSOS_t^{(r)}\right|$, where $PIN_t^{(r)}$ and $PSOS_t^{(r)}$ are the true values and $\widehat{PIN}_t^{(r)}$ and $\widehat{PSOS}_t^{(r)}$ denote the estimates based on the candidate estimation approach. The upper part plots the *MAE* of daily *PIN* estimates for each replication and the lower part plots the *MAE* of daily *PSOS* estimates.

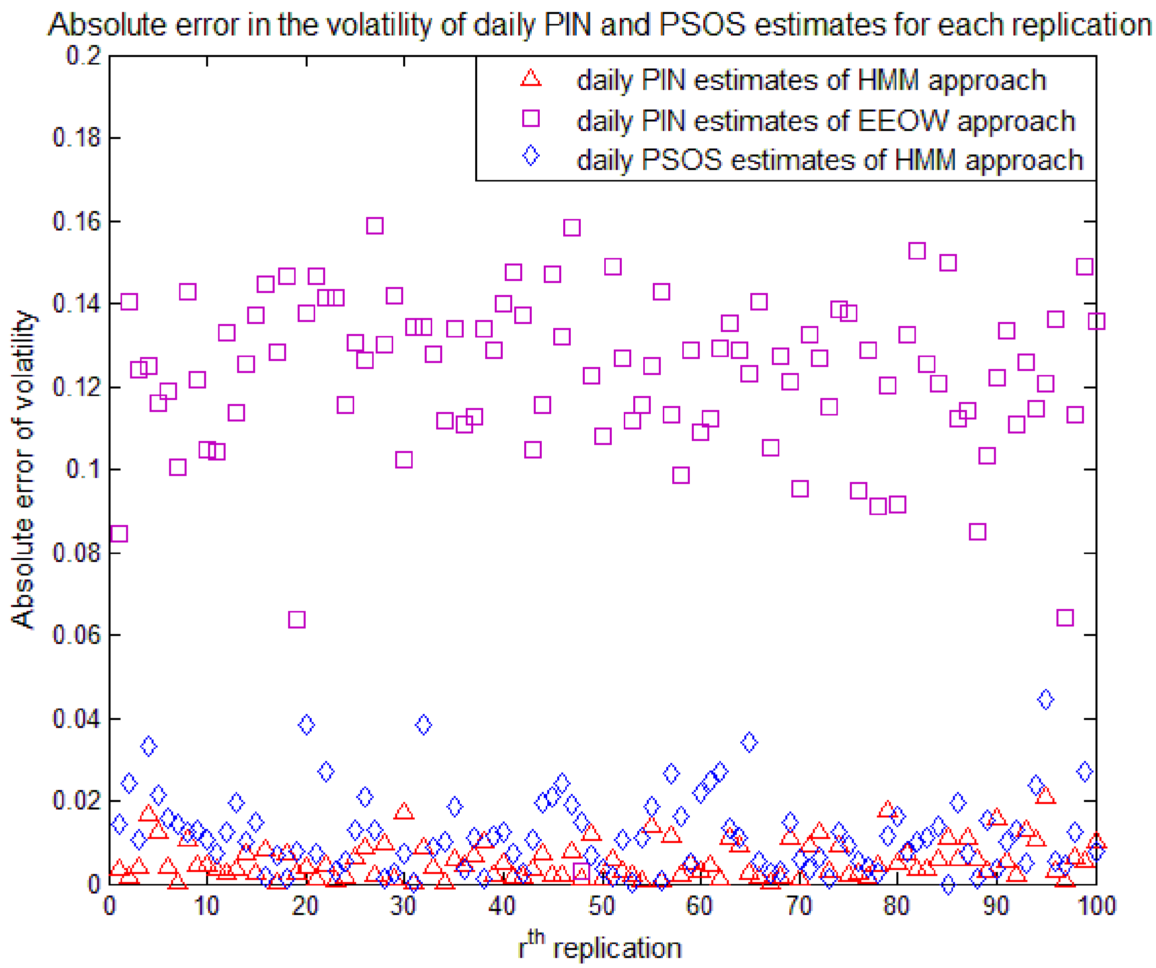

Figure S-5. Absolute error of standard deviation of daily *PIN* and *PSOS* estimates in each replication. The hypothetical trading data of 63 trading days are simulated according to the DY model with parameters $\alpha = 0.28$, $\delta = 0.3$, $\mu_b = 132$, $\mu_s = 133$, $\varepsilon_b = 121$, $\varepsilon_s = 123$, $\theta = \theta' = 0.1$, $\upsilon_b = 139$, $\upsilon_s = 131$ (Scenario 2.1 of Table S-I). The standard deviations (SD) of daily *PIN* and *PSOS* estimates in the r^{th} replication are computed respectively by $SD_{PIN} = \sqrt{\frac{1}{T}\sum_{t=1}^{T}\left(\widehat{PIN}_t^{(r)} - \frac{1}{T}\sum_{t=1}^{T}\widehat{PIN}_t^{(r)}\right)^2}$, $SD_{PSOS} = \sqrt{\frac{1}{T}\sum_{t=1}^{T}\left(\widehat{PSOS}_t^{(r)} - \frac{1}{T}\sum_{t=1}^{T}\widehat{PSOS}_t^{(r)}\right)^2}$, where $\widehat{PIN}_t^{(r)}$ and $\widehat{PSOS}_t^{(r)}$ denote the daily estimates of the HMM or EEOW approach. The absolute error is computed based on comparing the volatility of the daily estimates of the candidate approaches with the sample volatility of the true daily measures.

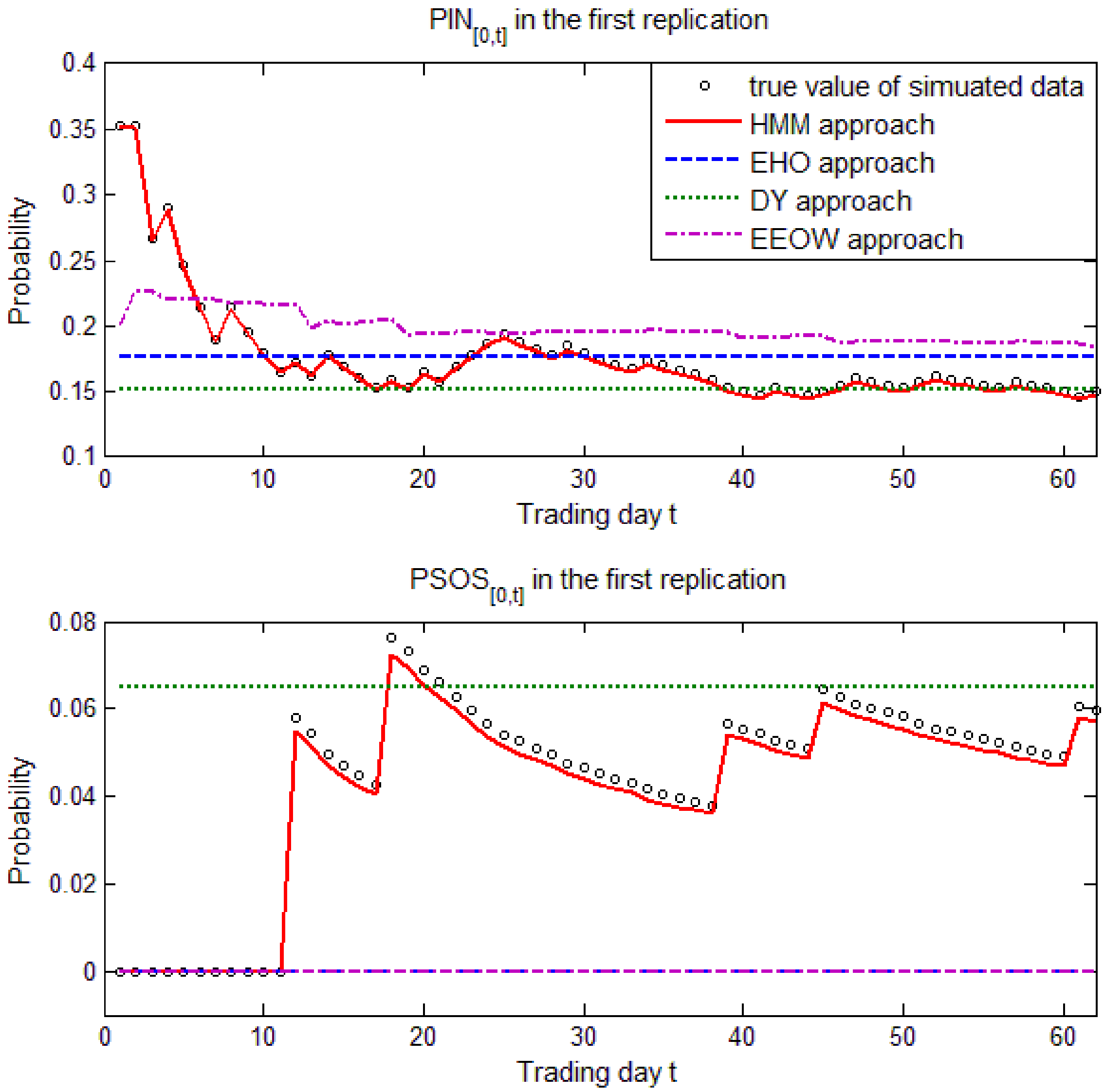

Figure S-6. Estimated *PIN* and *PSOS* over a time interval $[0,t]$ for $t = 1, 2, \dots, 63$. The hypothetical trading data of 63 trading days are simulated according to the DY model with parameters $\alpha = 0.28$, $\delta = 0.3$, $\mu_b = 132$, $\mu_s = 133$, $\varepsilon_b = 121$, $\varepsilon_s = 123$, $\theta = \theta' = 0.1$, $\upsilon_b = 139$, $\upsilon_s = 131$ (Scenario 2.1 of Table S-I). The upper and lower parts plot the cumulative *PIN* and *PSOS* estimates respectively.

Table S-I
Simulation results of state identification by the HMM

The hypothetical trading data of Scenarios 1.1 to 1.4, 2.1 to 2.4, 3.1 to 3.4, and 4.1 to 4.4 are generated based on the EHO, DY, EEOW, and extended EEOW models, respectively, over 63 or 252 trading days with 100 replications. For each replication, the mode of the conditional likelihood of hidden state for each trading day is compared with its true state realization so that we can count for misclassification rate over the whole estimation of 63 or 252 trading days. Panel A of this table reports the average misclassification rate of the 100 replications. Panel B reports the percentage of replications with initial hidden state correctly identified. The parameters of each simulation scenario are detailed in Panel C.

Panel A: Average misclassification rate over 100 replications

	$T = 63$		$T = 252$	
	AIC	BIC	AIC	BIC
Scenario 1.1	1.84%	2.04%	0.77%	0.72%
Scenario 1.2	3.09%	2.93%	0.41%	0.15%
Scenario 1.3	2.17%	1.98%	0.24%	0%
Scenario 1.4	3.77%	2.92%	0.38%	0.01%
Scenario 2.1	0.98%	1.55%	0.6%	0.82%
Scenario 2.2	5.41%	7.33%	2.13%	4.13%
Scenario 2.3	2.22%	2.38%	0.75%	0.91%
Scenario 2.4	4.61%	6.94%	1.38%	2.06%
Scenario 3.1	4.57%	2.95%	1.27%	0.84%
Scenario 3.2	3.47%	3.13%	1.55%	1.05%
Scenario 3.3	2.01%	1.14%	0.76%	0.17%
Scenario 3.4	1.86%	0.1%	0.23%	0.06%
Scenario 4.1	4.89%	4.48%	2.53%	2.14%
Scenario 4.2	7.92%	8.60%	6.54%	6.92%
Scenario 4.3	7.25%	9.46%	3.43%	5.23%
Scenario 4.4	7.05%	7.86%	2.14%	4.29%

Panel B: Percentage of replications with initial hidden state correctly identified

	$T = 63$		$T = 252$	
	AIC	BIC	AIC	BIC
Scenario 1.1	96%	99%	96%	100%
Scenario 1.2	98%	100%	98%	100%
Scenario 1.3	100%	100%	99%	99%
Scenario 1.4	99%	100%	100%	100%
Scenario 2.1	96%	95%	94%	94%
Scenario 2.2	85%	81%	87%	87%
Scenario 2.3	90%	91%	96%	97%
Scenario 2.4	89%	89%	93%	93%
Scenario 3.1	92%	94%	90%	95%
Scenario 3.2	95%	98%	98%	98%
Scenario 3.3	81%	82%	82%	82%
Scenario 3.4	86%	85%	87%	86%
Scenario 4.1	89%	88%	87%	86%
Scenario 4.2	94%	91%	99%	97%
Scenario 4.3	93%	90%	95%	91%
Scenario 4.4	83%	82%	83%	83%

Table S-I-*Continued*

Panel C: Parameters of each simulation scenario
Scenario 1.1: $\alpha = 0.28, \delta = 0.33, \mu = 31, \varepsilon_b = 23, \varepsilon_s = 24$
Scenario 1.2: $\alpha = 0.2, \delta = 0.65, \mu = 45, \varepsilon_b = 30, \varepsilon_s = 31$
Scenario 1.3: $\alpha = 0.35, \delta = 0.5, \mu = 192, \varepsilon_b = 205, \varepsilon_s = 205$
Scenario 1.4: $\alpha = 0.4, \delta = 0.5, \mu = 100, \varepsilon_b = 62, \varepsilon_s = 62$
Scenario 2.1: $\alpha = 0.28, \delta = 0.3, \mu_b = 132, \mu_s = 133, \varepsilon_b = 121, \varepsilon_s = 123, \theta = \theta' = 0.1, \upsilon_b = 139, \upsilon_s = 131$
Scenario 2.2: $\alpha = 0.45, \delta = 0.6, \mu_b = 32, \mu_s = 33, \varepsilon_b = 21, \varepsilon_s = 23, \theta = \theta' = 0.15, \upsilon_b = 39, \upsilon_s = 31$
Scenario 2.3: $\alpha = 0.4, \delta = 0.56, \mu_b = 240, \mu_s = 250, \varepsilon_b = 100, \varepsilon_s = 100, \theta = \theta' = 0.2, \upsilon_b = 130, \upsilon_s = 120$
Scenario 2.4: $\alpha = 0.3, \delta = 0.5, \mu_b = 50, \mu_s = 50, \varepsilon_b = 40, \varepsilon_s = 40, \theta = \theta' = 0.3, \upsilon_b = 80, \upsilon_s = 80$
Scenario 3.1: $\alpha = 0.3, \delta = 0.5, g = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \omega = \begin{pmatrix} 3 \\ 10 \end{pmatrix}, \Phi = \begin{pmatrix} 0.22 & 0.15 \\ -0.35 & 0.13 \end{pmatrix}, \Psi = \begin{pmatrix} 0.12 & 0.09 \\ 0.06 & 0.10 \end{pmatrix}, \begin{pmatrix} \alpha\mu_0 \\ 2\varepsilon_0 \end{pmatrix} = \begin{pmatrix} 8 \\ 16 \end{pmatrix}$
Scenario 3.2: $\alpha = 0.2, \delta = 0.4, g = \begin{pmatrix} 4E^{-5} \\ E^{-5} \end{pmatrix}, \omega = \begin{pmatrix} 4 \\ 12 \end{pmatrix}, \Phi = \begin{pmatrix} 0.33 & -0.1 \\ -0.02 & 0.01 \end{pmatrix}, \Psi = \begin{pmatrix} 0.14 & 0.12 \\ 0.18 & 0.18 \end{pmatrix}, \begin{pmatrix} \alpha\mu_0 \\ 2\varepsilon_0 \end{pmatrix} = \begin{pmatrix} 4 \\ 13 \end{pmatrix}$
Scenario 3.3: $\alpha = 0.4, \delta = 0.6, g = \begin{pmatrix} E^{-5} \\ 0 \end{pmatrix}, \omega = \begin{pmatrix} 3 \\ 69 \end{pmatrix}, \Phi = \begin{pmatrix} -0.08 & 0.36 \\ 0.01 & -0.02 \end{pmatrix}, \Psi = \begin{pmatrix} 0.13 & 0.10 \\ -0.19 & 0.20 \end{pmatrix}, \begin{pmatrix} \alpha\mu_0 \\ 2\varepsilon_0 \end{pmatrix} = \begin{pmatrix} 2 \\ 69 \end{pmatrix}$
Scenario 3.4: $\alpha = 0.2, \delta = 0.55, g = \begin{pmatrix} 0 \\ 2E^{-5} \end{pmatrix}, \omega = \begin{pmatrix} 15 \\ 48 \end{pmatrix}, \Phi = \begin{pmatrix} -0.01 & 0.03 \\ -0.07 & 0.02 \end{pmatrix}, \Psi = \begin{pmatrix} 0.08 & 0.07 \\ 0.03 & 0.03 \end{pmatrix}, \begin{pmatrix} \alpha\mu_0 \\ 2\varepsilon_0 \end{pmatrix} = \begin{pmatrix} 6 \\ 40 \end{pmatrix}$
Scenario 4.1: $\alpha = 0.2, \delta = 0.5, \theta = \theta' = 0.2, \varepsilon_t = 50, g = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \omega = \begin{pmatrix} 12 \\ 112 \end{pmatrix}, \Phi = \begin{pmatrix} 0.03 & 0.02 \\ 0.04 & 0.28 \end{pmatrix},$ $\Psi = \begin{pmatrix} 0.08 & 0.06 \\ -0.01 & 0.01 \end{pmatrix}, \begin{pmatrix} \alpha\mu_0 \\ 2\varepsilon_0 + 2\theta\upsilon_0 \end{pmatrix} = \begin{pmatrix} 12 \\ 113 \end{pmatrix}$
Scenario 4.2: $\alpha = 0.3, \delta = 0.4, \theta = \theta' = 0.3, \varepsilon_t = 40, g = \begin{pmatrix} 2E^{-5} \\ E^{-5} \end{pmatrix}, \omega = \begin{pmatrix} 40 \\ 28 \end{pmatrix}, \Phi = \begin{pmatrix} 0.13 & -0.19 \\ -0.26 & 0.20 \end{pmatrix},$ $\Psi = \begin{pmatrix} 0.09 & 0.06 \\ 0.23 & 0.3 \end{pmatrix}, \begin{pmatrix} \alpha\mu_0 \\ 2\varepsilon_0 + 2\theta\upsilon_0 \end{pmatrix} = \begin{pmatrix} 40 \\ 60 \end{pmatrix}$
Scenario 4.3: $\alpha = 0.4, \delta = 0.5, \theta = \theta' = 0.2, \upsilon_t = 40, g = \begin{pmatrix} E^{-5} \\ E^{-5} \end{pmatrix}, \omega = \begin{pmatrix} 18 \\ 38 \end{pmatrix}, \Phi = \begin{pmatrix} 0.07 & -0.04 \\ -0.28 & 0.29 \end{pmatrix},$ $\Psi = \begin{pmatrix} 0.05 & 0.07 \\ 0.25 & 0.22 \end{pmatrix}, \begin{pmatrix} \alpha\mu_0 \\ 2\varepsilon_0 + 2\theta\upsilon_0 \end{pmatrix} = \begin{pmatrix} 20 \\ 40 \end{pmatrix}$
Scenario 4.4: $\alpha = 0.45, \delta = 0.6, \theta = \theta' = 0.15, \upsilon_t = 10, g = \begin{pmatrix} E^{-5} \\ 2E^{-5} \end{pmatrix}, \omega = \begin{pmatrix} 2 \\ 15 \end{pmatrix}, \Phi = \begin{pmatrix} 0.31 & -0.13 \\ -0.54 & 0 \end{pmatrix},$ $\Psi = \begin{pmatrix} 0.19 & 0.18 \\ 0.26 & 0.22 \end{pmatrix}, \begin{pmatrix} \alpha\mu_0 \\ 2\varepsilon_0 + 2\theta\upsilon_0 \end{pmatrix} = \begin{pmatrix} 3 \\ 16 \end{pmatrix}$

Table S-II

Simulation results of daily estimates of *PIN* and *PSOS*

The hypothetical trading data of Scenarios 1.1 to 1.4, 2.1 to 2.4 3.1 to 3.4, and 4.1 to 4.4 are generated based on the EHO, DY, EEOW, and extended EEOW models, respectively, over 63 or 252 trading days. The mean absolute error (*MAE*) is given by

$$MAE^{(r)}_{PIN_t^{[0,T]}} = \frac{1}{T}\sum_{t=1}^{T}\left|\widehat{PIN}_t^{(r)} - PIN_t^{(r)}\right|, \qquad MAE^{(r)}_{PSOS_t^{[0,T]}} = \frac{1}{T}\sum_{t=1}^{T}\left|\widehat{PSOS}_t^{(r)} - PSOS_t^{(r)}\right|,$$

where $PIN_t^{(r)}$ and $PSOS_t^{(r)}$ are respectively the true daily *PIN* and *PSOS* on day *t* in the r^{th} replication, and $\widehat{PIN}_t^{(r)}$ and $\widehat{PSOS}_t^{(r)}$ are daily estimates in the r^{th} replication obtained by using the candidate approach. Over 100 replications, we report the mean of *MAE* defined by

$$Mean\left(MAE^{(r)}_{PIN_t^{[0,T]}}\right) = \frac{1}{100}\sum_{r=1}^{100} MAE^{(r)}_{PIN_t^{[0,T]}}, \qquad Mean\left(MAE^{(r)}_{PSOS_t^{[0,T]}}\right) = \frac{1}{100}\sum_{r=1}^{100} MAE^{(r)}_{PSOS_t^{[0,T]}}.$$

The mean of the standard deviation of daily *PIN* or *PSOS* estimates over the 100 replications is,

$$Mean\left(SD^{(r)}_{PIN_t^{[0,T]}}\right) = \frac{1}{100}\sum_{r=1}^{100}\sqrt{\frac{1}{T}\sum_{t=1}^{T}\left(\widehat{PIN}_t^{(r)} - \frac{1}{T}\sum_{t=1}^{T}\widehat{PIN}_t^{(r)}\right)^2},$$

$$Mean\left(SD^{(r)}_{PSOS_t^{[0,T]}}\right) = \frac{1}{100}\sum_{r=1}^{100}\sqrt{\frac{1}{T}\sum_{t=1}^{T}\left(\widehat{PSOS}_t^{(r)} - \frac{1}{T}\sum_{t=1}^{T}\widehat{PSOS}_t^{(r)}\right)^2}.$$

Each row is associated with a candidate approach. The true values are given in the last row of each scenario denoted by TRUE.

	$Mean\left(MAE^{(r)}_{PIN_t^{[0,T]}}\right)$		$Mean\left(MAE^{(r)}_{PSOS_t^{[0,T]}}\right)$		$Mean\left(SD^{(r)}_{PIN_t^{[0,T]}}\right)$		$Mean\left(SD^{(r)}_{PSOS_t^{[0,T]}}\right)$	
	$T = 63$	$T = 252$	$T = 63$	$T = 252$	$T = 63$	$T = 252$	$T = 63$	$T = 252$
Scenario 1.1: Trading data generated by the EHO model of $\alpha = 0.28, \delta = 0.33, \mu = 31, \varepsilon_b = 23, \varepsilon_s = 24$								
HMM	0.0182	0.0103	0.0000	0.0000	0.1687	0.1718	0.0000	0.0000
EHO	0.1758	0.1775	0	0	0	0	0	0
DY	0.1752	0.1772	0.0000	0.0000	0	0	0	0
EEOW	0.1739	0.1775	0	0	0.0373	0.0128	0	0
TRUE	0	0	0	0	0.1762	0.1732	0	0
Scenario 1.2: Trading data generated by the EHO model of $\alpha = 0.2, \delta = 0.65, \mu = 45, \varepsilon_b = 30, \varepsilon_s = 31$								
HMM	0.0133	0.0074	0.0000	0.0000	0.1654	0.1669	0.0000	0.0000
EHO	0.1574	0.1602	0	0	0	0	0	0
DY	0.1567	0.1599	0.0021	0.0010	0	0	0	0
EEOW	0.1683	0.1635	0	0	0.0376	0.0136	0	0
TRUE	0	0	0	0	0.1706	0.169	0	0
Scenario 1.3: Trading data generated by the EHO model of $\alpha = 0.35, \delta = 0.5, \mu = 192, \varepsilon_b = 205, \varepsilon_s = 205$								
HMM	0.0069	0.0028	0.0000	0.0000	0.1495	0.1506	0.0000	0.0000
EHO	0.1515	0.1531	0	0	0	0	0	0
DY	0.1515	0.1531	0.0000	0.0000	0	0	0	0
EEOW	0.1547	0.1548	0	0	0.013	0.0057	0	0
TRUE	0	0	0	0	0.1521	0.152	0	0
Scenario 1.4: Trading data generated by the EHO model of $\alpha = 0.4, \delta = 0.5, \mu = 100, \varepsilon_b = 62, \varepsilon_s = 62$								
HMM	0.0094	0.0041	0.0000	0.0000	0.2157	0.2173	0.0000	0.0000
EHO	0.2235	0.2264	0	0	0	0	0	0
DY	0.2235	0.2264	0.0000	0.0000	0	0	0	0
EEOW	0.2263	0.2265	0	0	0.0203	0.0073	0	0
TRUE	0	0	0	0	0.2186	0.2188	0	0
Scenario 2.1: Trading data generated by the DY model of $\alpha = 0.28, \delta = 0.3, \mu_b = 132, \mu_s = 133, \varepsilon_b = 121, \varepsilon_s = 123, \theta = \theta' = 0.1, \upsilon_b = 139, \upsilon_s = 131$								
HMM	0.0107	0.0065	0.0068	0.0044	0.1484	0.1522	0.1385	0.1404
EHO	0.1602	0.1622	0.0502	0.0499	0	0	0	0
DY	0.143	0.1454	0.1184	0.1202	0	0	0	0
EEOW	0.1719	0.171	0.0502	0.0499	0.0358	0.0163	0	0
TRUE	0	0	0	0	0.1518	0.153	0.1478	0.1467

Table S-II-*Continued*

	$Mean\left(MAE^{(r)}_{PIN^{[0,T]}_t}\right)$		$Mean\left(MAE^{(r)}_{PSOS^{[0,T]}_t}\right)$		$Mean\left(SD^{(r)}_{PIN^{[0,T]}_t}\right)$		$Mean\left(SD^{(r)}_{PSOS^{[0,T]}_t}\right)$	
	$T = 63$	$T = 252$	$T = 63$	$T = 252$	$T = 63$	$T = 252$	$T = 63$	$T = 252$
Scenario 2.2: Trading data generated by the DY model of $\alpha = 0.45, \delta = 0.6, \mu_b = 32, \mu_s = 33, \varepsilon_b = 21, \varepsilon_s = 23, \theta = \theta' = 0.15, \upsilon_b = 39, \upsilon_s = 31$								
HMM	0.0226	0.015	0.0183	0.0166	0.195	0.1933	0.175	0.1793
EHO	0.2072	0.2098	0.0817	0.084	0	0	0	0
DY	0.1956	0.1976	0.1843	0.1897	0	0	0	0
EEOW	0.2156	0.2171	0.0817	0.084	0.067	0.0393	0	0
TRUE	0	0	0	0	0.2017	0.2016	0.1961	0.197
Scenario 2.3: Trading data generated by the DY model of $\alpha = 0.4, \delta = 0.56, \mu_b = 240, \mu_s = 250, \varepsilon_b = 100, \varepsilon_s = 100, \theta = \theta' = 0.2, \upsilon_b = 130, \upsilon_s = 120$								
HMM	0.0126	0.0075	0.0143	0.0166	0.2517	0.2519	0.1815	0.185
EHO	0.2698	0.2702	0.1778	0.1817	0	0	0	0
DY	0.2577	0.2592	0.0943	0.0966	0	0	0	0
EEOW	0.2638	0.2677	0	0	0.0508	0.0234	0	0
TRUE	0	0	0	0	0.2552	0.2549	0.1948	0.1959
Scenario 2.4: Trading data generated by the DY model of $\alpha = 0.3, \delta = 0.5, \mu_b = 50, \mu_s = 50, \varepsilon_b = 40, \varepsilon_s = 40, \theta = \theta' = 0.3, \upsilon_b = 80, \upsilon_s = 80$								
HMM	0.0193	0.0084	0.022	0.0073	0.1487	0.1546	0.2691	0.283
EHO	0.1925	0.1912	0.1901	0.187	0	0	0	0
DY	0.1341	0.1381	0.3176	0.3217	0	0	0	0
EEOW	0.1964	0.1992	0.1901	0.187	0.0947	0.0364	0	0
TRUE	0	0	0	0	0.1553	0.1569	0.2909	0.29
Scenario 3.1: Trading data generated by the EEOW model of $\alpha = 0.3, \delta = 0.5, g = \begin{pmatrix}0\\0\end{pmatrix}, \omega = \begin{pmatrix}3\\10\end{pmatrix}, \Phi = \begin{pmatrix}0.22 & 0.15\\-0.35 & 0.13\end{pmatrix}, \Psi = \begin{pmatrix}0.12 & 0.09\\0.06 & 0.10\end{pmatrix}, \begin{pmatrix}\alpha\mu_0\\2\varepsilon_0\end{pmatrix} = \begin{pmatrix}8\\16\end{pmatrix}$								
HMM	0.0197	0.0114	0.0000	0.0000	0.326	0.3336	0.0000	0.0000
EHO	0.3679	0.3959	0	0	0	0	0	0
DY	0.3448	0.3802	0.0195	0.0338	0	0	0	0
EEOW	0.3448	0.3987	0	0	0.066	0.0422	0	0
TRUE	0	0	0	0	0.3289	0.3351	0	0
Scenario 3.2: Trading data generated by the EEOW model of $\alpha = 0.2, \delta = 0.4, g = \begin{pmatrix}4E^{-5}\\E^{-5}\end{pmatrix}, \omega = \begin{pmatrix}4\\12\end{pmatrix}, \Phi = \begin{pmatrix}0.33 & -0.1\\-0.02 & 0.01\end{pmatrix}, \Psi = \begin{pmatrix}0.14 & 0.12\\0.18 & 0.18\end{pmatrix}, \begin{pmatrix}\alpha\mu_0\\2\varepsilon_0\end{pmatrix} = \begin{pmatrix}4\\13\end{pmatrix}$								
HMM	0.0149	0.0028	0.0000	0.0000	0.2764	0.2863	0.0000	0.0000
EHO	0.312	0.3466	0	0	0	0	0	0
DY	0.2123	0.2571	0.0354	0.051	0	0	0	0
EEOW	0.3335	0.3406	0	0	0.0576	0.0301	0	0
TRUE	0	0	0	0	0.2828	0.2856	0	0
Scenario 3.3: Trading data generated by the EEOW model of $\alpha = 0.4, \delta = 0.6, g = \begin{pmatrix}E^{-5}\\0\end{pmatrix}, \omega = \begin{pmatrix}3\\69\end{pmatrix}, \Phi = \begin{pmatrix}-0.08 & 0.36\\0.01 & -0.02\end{pmatrix}, \Psi = \begin{pmatrix}0.13 & 0.10\\-0.19 & 0.20\end{pmatrix}, \begin{pmatrix}\alpha\mu_0\\2\varepsilon_0\end{pmatrix} = \begin{pmatrix}2\\69\end{pmatrix}$								
HMM	0.0253	0.0052	0.0000	0.0000	0.2778	0.2789	0.0000	0.0000
EHO	0.2762	0.278	0	0	0	0	0	0
DY	0.2907	0.2934	0.0005	0.0003	0	0	0	0
EEOW	0.291	0.2951	0	0	0.0667	0.0664	0	0
TRUE	0	0	0	0	0.2818	0.2798	0	0

Table S-II-*Continued*

	$Mean\left(MAE^{(r)}_{PIN_t^{[0,T]}}\right)$		$Mean\left(MAE^{(r)}_{PSOS_t^{[0,T]}}\right)$		$Mean\left(SD^{(r)}_{PIN_t^{[0,T]}}\right)$		$Mean\left(SD^{(r)}_{PSOS_t^{[0,T]}}\right)$	
	$T=63$	$T=252$	$T=63$	$T=252$	$T=63$	$T=252$	$T=63$	$T=252$
Scenario 3.4: Trading data generated by the EEOW model of $\alpha=0.2, \delta=0.55, g=\begin{pmatrix}0\\2E^{-5}\end{pmatrix}, \omega=\begin{pmatrix}15\\48\end{pmatrix}, \Phi=\begin{pmatrix}-0.01 & 0.03\\-0.07 & 0.02\end{pmatrix}, \Psi=\begin{pmatrix}0.08 & 0.07\\0.03 & 0.03\end{pmatrix}, \begin{pmatrix}\alpha\mu_0\\2\varepsilon_0\end{pmatrix}=\begin{pmatrix}6\\40\end{pmatrix}$								
HMM	0.0138	0.0054	0.0000	0.0000	0.2659	0.2703	0.0000	0.0000
EHO	0.2623	0.2714	0	0	0	0	0	0
DY	0.2947	0.3073	0.0112	0.01	0	0	0	0
EEOW	0.3112	0.312	0	0	0.0411	0.0281	0	0
TRUE	0	0	0	0	0.2699	0.2705	0	0
Scenario 4.1: Trading data generated by extended EEOW model of $\alpha=0.2, \delta=0.5, \theta=\theta'=0.2, \varepsilon_t=50, g=\begin{pmatrix}0\\0\end{pmatrix}, \omega=\begin{pmatrix}12\\112\end{pmatrix}, \Phi=\begin{pmatrix}0.03 & 0.02\\0.04 & 0.28\end{pmatrix}, \Psi=\begin{pmatrix}0.08 & 0.06\\-0.01 & 0.01\end{pmatrix}, \begin{pmatrix}\alpha\mu_0\\2\varepsilon_0+2\theta\upsilon_0\end{pmatrix}=\begin{pmatrix}12\\113\end{pmatrix}$								
HMM	0.0193	0.0099	0.0106	0.0048	0.2017	0.2127	0.2840	0.2713
EHO	0.2334	0.243	0.1337	0.1421	0	0	0	0
DY	0.1888	0.1912	0.3093	0.3279	0	0	0	0
EEOW	0.2687	0.2732	0.1337	0.1421	0.0734	0.0437	0	0
TRUE	0	0	0	0	0.2083	0.2133	0.2792	0.2852
Scenario 4.2: Trading data generated by extended EEOW model of $\alpha=0.3, \delta=0.4, \theta=\theta'=0.3, \varepsilon_t=40, g=\begin{pmatrix}2E^{-5}\\E^{-5}\end{pmatrix}, \omega=\begin{pmatrix}40\\28\end{pmatrix}, \Phi=\begin{pmatrix}0.13 & -0.19\\-0.26 & 0.20\end{pmatrix}, \Psi=\begin{pmatrix}0.09 & 0.06\\0.23 & 0.3\end{pmatrix}, \begin{pmatrix}\alpha\mu_0\\2\varepsilon_0+2\theta\upsilon_0\end{pmatrix}=\begin{pmatrix}40\\60\end{pmatrix}$								
HMM	0.0235	0.0144	0.0384	0.0338	0.2663	0.2734	0.1920	0.1840
EHO	0.265	0.2691	0.1010	0.1011	0	0	0	0
DY	0.2922	0.2967	0.1535	0.1551	0	0	0	0
EEOW	0.2981	0.3023	0.1010	0.1011	0.0618	0.0454	0	0
TRUE	0	0	0	0	0.2751	0.2758	0.1827	0.1829
Scenario 4.3: Trading data generated by extended EEOW model of $\alpha=0.4, \delta=0.5, \theta=\theta'=0.2, \upsilon_t=40, g=\begin{pmatrix}E^{-5}\\E^{-5}\end{pmatrix}, \omega=\begin{pmatrix}18\\38\end{pmatrix}, \Phi=\begin{pmatrix}0.07 & -0.04\\-0.28 & 0.29\end{pmatrix}, \Psi=\begin{pmatrix}0.05 & 0.07\\0.25 & 0.22\end{pmatrix}, \begin{pmatrix}\alpha\mu_0\\2\varepsilon_0+2\theta\upsilon_0\end{pmatrix}=\begin{pmatrix}20\\40\end{pmatrix}$								
HMM	0.0255	0.016	0.0362	0.0269	0.2162	0.2232	0.2153	0.2152
EHO	0.2293	0.2312	0.1047	0.1012	0	0	0	0
DY	0.2229	0.2254	0.2060	0.2042	0	0	0	0
EEOW	0.2381	0.2391	0.1047	0.1012	0.0506	0.0241	0	0
TRUE	0	0	0	0	0.2264	0.227	0.2279	0.2268
Scenario 4.4: Trading data generated by extended EEOW model of $\alpha=0.45, \delta=0.6, \theta=\theta'=0.15, \upsilon_t=10, g=\begin{pmatrix}E^{-5}\\2E^{-5}\end{pmatrix}, \omega=\begin{pmatrix}2\\15\end{pmatrix}, \Phi=\begin{pmatrix}0.31 & -0.13\\-0.54 & 0\end{pmatrix}, \Psi=\begin{pmatrix}0.19 & 0.18\\0.26 & 0.22\end{pmatrix}, \begin{pmatrix}\alpha\mu_0\\2\varepsilon_0+2\theta\upsilon_0\end{pmatrix}=\begin{pmatrix}3\\16\end{pmatrix}$								
HMM	0.0237	0.0119	0.0328	0.0275	0.2115	0.2188	0.1925	0.1918
EHO	0.2159	0.2217	0.0777	0.0764	0	0	0	0
DY	0.2121	0.2143	0.1938	0.1898	0	0	0	0
EEOW	0.2222	0.221	0.0777	0.0764	0.0704	0.0435	0	0
TRUE	0	0	0	0	0.2227	0.2221	0.1874	0.1867

Table S-III

Simulation results of estimated *PIN* and *PSOS* over a certain time interval

The hypothetical trading data of Scenarios 1.1 to 1.4, 2.1 to 2.4, 3.1 to 3.4, and 4.1 to 4.4 are generated based on the EHO, DY, EEOW, and extended EEOW models, respectively, over 252 trading days. For 100 replications, the bias (*BIAS*) between the estimates and the corresponding true values over the interval of day 0 through day τ are calcualted by

$$BIAS_{PIN_{[0,\tau]}} = \frac{1}{100}\sum_{r=1}^{100}\left(\widehat{PIN}_{[0,\tau]}^{(r)} - PIN_{[0,\tau]}^{(r)}\right), \qquad BIAS_{PSOS_{[0,\tau]}} = \frac{1}{100}\sum_{r=1}^{100}\left(\widehat{PSOS}_{[0,\tau]}^{(r)} - PSOS_{[0,\tau]}^{(r)}\right),$$

where $PIN_{[0,\tau]}^{(r)}$ and $PSOS_{[0,\tau]}^{(r)}$ are respectively the true *PIN* and *PSOS* over the period $[0,\tau]$ in the r^{th} replication and $\widehat{PIN}_{[0,\tau]}^{(r)}$ and $\widehat{PSOS}_{[0,\tau]}^{(r)}$ are the estimates obtained by a candidate approach. The corresponding root mean squared errors (*RMSE*) reported in parentheses are calcualted by

$$RMSE_{PIN_{[0,\tau]}} = \left\{\frac{1}{100}\sum_{r=1}^{100}\left[\widehat{PIN}_{[0,\tau]}^{(r)} - PIN_{[0,\tau]}^{(r)}\right]^2\right\}^{\frac{1}{2}}, \qquad RMSE_{PSOS_{[0,\tau]}} = \left\{\frac{1}{100}\sum_{r=1}^{100}\left[\widehat{PSOS}_{[0,\tau]}^{(r)} - PSOS_{[0,\tau]}^{(r)}\right]^2\right\}^{\frac{1}{2}}.$$

	$BIAS_{PIN_{[0,\tau]}}$ $\left(RMSE_{PIN_{[0,\tau]}}\right)$				$BIAS_{PSOS_{[0,\tau]}}$ $\left(RMSE_{PSOS_{[0,\tau]}}\right)$			
	$\tau = 5$	$\tau = 21$	$\tau = 63$	$\tau = 252$	$\tau = 5$	$\tau = 21$	$\tau = 63$	$\tau = 252$
Scenario 1.1: Trading data generated by the EHO model of $\alpha = 0.28, \delta = 0.33, \mu = 31, \varepsilon_b = 23, \varepsilon_s = 24$								
HMM	0.01	0.0065	0.0056	0.0047	0.0000	0.0000	0.0000	0.0000
	(0.0166)	(0.0091)	(0.0074)	(0.0061)	(0.0000)	(0.0000)	(0.0000)	(0.0000)
EHO	0.0727	0.0365	0.0204	0.004	0	0	0	0
	(0.0881)	(0.0435)	(0.0261)	(0.0049)	(0)	(0)	(0)	(0)
DY	0.0727	0.0365	0.0204	0.0042	0.0011	0.0011	0.0011	0.0011
	(0.0883)	(0.0434)	(0.0262)	(0.0052)	(0.0026)	(0.0026)	(0.0026)	(0.0026)
EEOW	0.0723	0.0362	0.0215	0.0047	0	0	0	0
	(0.0884)	(0.0435)	(0.027)	(0.0061)	(0)	(0)	(0)	(0)
Scenario 1.2: Trading data generated by the EHO model of $\alpha = 0.2, \delta = 0.65, \mu = 45, \varepsilon_b = 30, \varepsilon_s = 31$								
HMM	0.0054	0.0056	0.0049	0.0046	0.0000	0.0000	0.0000	0.0000
	(0.0143)	(0.0082)	(0.0072)	(0.0066)	(0.0000)	(0.0000)	(0.0000)	(0.0000)
EHO	0.0865	0.0351	0.0195	0.0028	0	0	0	0
	(0.1021)	(0.0442)	(0.0235)	(0.0034)	(0)	(0)	(0)	(0)
DY	0.0859	0.0348	0.0187	0.003	0.0006	0.0006	0.0006	0.0006
	(0.1019)	(0.0441)	(0.0231)	(0.0037)	(0.0022)	(0.0022)	(0.0022)	(0.0022)
EEOW	0.0904	0.0419	0.0264	0.0083	0	0	0	0
	(0.1129)	(0.0648)	(0.0463)	(0.0254)	(0)	(0)	(0)	(0)
Scenario 1.3: Trading data generated by the EHO model of $\alpha = 0.35, \delta = 0.5, \mu = 192, \varepsilon_b = 205, \varepsilon_s = 205$								
HMM	0.0028	0.0026	0.0025	0.002	0.0000	0.0000	0.0000	0.0000
	(0.0047)	(0.0036)	(0.0034)	(0.0027)	(0.0000)	(0.0000)	(0.0000)	(0.0000)
EHO	0.0596	0.0264	0.0146	0.0013	0	0	0	0
	(0.0728)	(0.0325)	(0.0184)	(0.0017)	(0)	(0)	(0)	(0)
DY	0.0596	0.0264	0.0146	0.0013	0.0000	0.0000	0.0000	0.0000
	(0.0728)	(0.0325)	(0.0184)	(0.0017)	(0.0000)	(0.0000)	(0.0000)	(0.0000)
EEOW	0.0726	0.0334	0.0192	0.0063	0	0	0	0
	(0.0909)	(0.0423)	(0.0268)	(0.015)	(0)	(0)	(0)	(0)
Scenario 1.4: Trading data generated by the EHO model of $\alpha = 0.4, \delta = 0.5, \mu = 100, \varepsilon_b = 62, \varepsilon_s = 62$								
HMM	0.0044	0.0038	0.0039	0.0029	0.0000	0.0000	0.0000	0.0000
	(0.0057)	(0.0048)	(0.0048)	(0.0038)	(0.0000)	(0.0000)	(0.0000)	(0.0000)
EHO	0.0881	0.0405	0.02	0.0021	0	0	0	0
	(0.1061)	(0.05)	(0.0257)	(0.0027)	(0)	(0)	(0)	(0)
DY	0.0881	0.0405	0.02	0.0021	0.0000	0.0000	0.0000	0.0000
	(0.1061)	(0.05)	(0.0257)	(0.0027)	(0.0000)	(0.0000)	(0.0000)	(0.0000)
EEOW	0.085	0.041	0.0205	0.0032	0	0	0	0
	(0.1058)	(0.0503)	(0.0261)	(0.0045)	(0)	(0)	(0)	(0)

Table S-III-*Continued*

	$BIAS_{PIN_{[0,\tau]}}$ $\left(RMSE_{PIN_{[0,\tau]}}\right)$				$BIAS_{PSOS_{[0,\tau]}}$ $\left(RMSE_{PSOS_{[0,\tau]}}\right)$			
	$\tau = 5$	$\tau = 21$	$\tau = 63$	$\tau = 252$	$\tau = 5$	$\tau = 21$	$\tau = 63$	$\tau = 252$
Scenario 2.1: Trading data generated by the DY model of $\alpha = 0.28, \delta = 0.3, \mu_b = 132, \mu_s = 133, \varepsilon_b = 121, \varepsilon_s = 123, \theta = \theta' = 0.1, \upsilon_b = 139, \upsilon_s = 131$								
HMM	0.0122 (0.0208)	0.0144 (0.018)	0.0127 (0.0165)	0.01 (0.0131)	0.0199 (0.0172)	0.0108 (0.0174)	0.0107 (0.0142)	0.0096 (0.0106)
EHO	0.0903 (0.1163)	0.0586 (0.0692)	0.0567 (0.0617)	0.057 (0.0578)	0.136 (0.202)	0.1417 (0.1572)	0.149 (0.1539)	0.1492 (0.1504)
DY	0.079 (0.0961)	0.0322 (0.0436)	0.0193 (0.024)	0.004 (0.0049)	0.1352 (0.1481)	0.057 (0.0691)	0.0287 (0.036)	0.0036 (0.0048)
EEOW	0.0955 (0.1186)	0.0583 (0.0709)	0.0535 (0.0615)	0.049 (0.0519)	0.136 (0.202)	0.1417 (0.1572)	0.149 (0.1539)	0.1492 (0.1504)
Scenario 2.2: Trading data generated by the DY model of $\alpha = 0.45, \delta = 0.6, \mu_b = 32, \mu_s = 33, \varepsilon_b = 21, \varepsilon_s = 23, \theta = \theta' = 0.15, \upsilon_b = 39, \upsilon_s = 31$								
HMM	0.0073 (0.0136)	0.0048 (0.0075)	0.0042 (0.0057)	0.0031 (0.0043)	0.0042 (0.0092)	0.0044 (0.006)	0.0043 (0.0053)	0.0043 (0.0103)
EHO	0.0758 (0.0881)	0.045 (0.0575)	0.0392 (0.044)	0.0375 (0.0378)	0.078 (0.1313)	0.0863 (0.1025)	0.0846 (0.0894)	0.0885 (0.0894)
DY	0.063 (0.0801)	0.0297 (0.0399)	0.0163 (0.0204)	0.0014 (0.0018)	0.1015 (0.107)	0.0425 (0.0524)	0.0196 (0.0238)	0.0016 (0.0019)
EEOW	0.1028 (0.1296)	0.0688 (0.0985)	0.0591 (0.0924)	0.0535 (0.0602)	0.078 (0.1313)	0.0863 (0.1025)	0.0846 (0.0894)	0.0885 (0.0894)
Scenario 2.3: Trading data generated by the DY model of $\alpha = 0.4, \delta = 0.56, \mu_b = 240, \mu_s = 250, \varepsilon_b = 100, \varepsilon_s = 100, \theta = \theta' = 0.2, \upsilon_b = 130, \upsilon_s = 120$								
HMM	0.0067 (0.009)	0.0041 (0.0054)	0.0034 (0.0044)	0.0032 (0.0039)	0.008 (0.0127)	0.0076 (0.0097)	0.0077 (0.0099)	0.0081 (0.0106)
EHO	0.1141 (0.1495)	0.0681 (0.0813)	0.0533 (0.0609)	0.0542 (0.0546)	0.1331 (0.1789)	0.147 (0.157)	0.1435 (0.147)	0.1448 (0.1458)
DY	0.1038 (0.1339)	0.0472 (0.0597)	0.026 (0.0312)	0.0015 (0.0019)	0.0965 (0.1153)	0.0437 (0.0513)	0.0229 (0.028)	0.0018 (0.0022)
EEOW	0.1037 (0.1365)	0.062 (0.0757)	0.0496 (0.0582)	0.0464 (0.0491)	0.1331 (0.1789)	0.147 (0.157)	0.1435 (0.147)	0.1448 (0.1458)
Scenario 2.4: Trading data generated by the DY model of $\alpha = 0.3, \delta = 0.5, \mu_b = 50, \mu_s = 50, \varepsilon_b = 40, \varepsilon_s = 40, \theta = \theta' = 0.3, \upsilon_b = 80, \upsilon_s = 80$								
HMM	0.0081 (0.0127)	0.0048 (0.0067)	0.0045 (0.0059)	0.0041 (0.0052)	0.0089 (0.0131)	0.0086 (0.011)	0.0088 (0.0115)	0.0085 (0.011)
EHO	0.1087 (0.1278)	0.1104 (0.1163)	0.1118 (0.114)	0.111 (0.112)	0.2831 (0.3319)	0.3111 (0.3211)	0.3296 (0.3334)	0.3317 (0.3324)
DY	0.0711 (0.0877)	0.0268 (0.0334)	0.0135 (0.0177)	0.0025 (0.0031)	0.1361 (0.1776)	0.0606 (0.0785)	0.0361 (0.0449)	0.0035 (0.0043)
EEOW	0.1141 (0.1429)	0.1062 (0.118)	0.1149 (0.1231)	0.1166 (0.1287)	0.2831 (0.3319)	0.3111 (0.3211)	0.3296 (0.3334)	0.3317 (0.3324)

Table S-III-*Continued*

	$BIAS_{PIN_{[0,\tau]}}$ $(RMSE_{PIN_{[0,\tau]}})$				$BIAS_{PSOS_{[0,\tau]}}$ $(RMSE_{PSOS_{[0,\tau]}})$			
	$\tau = 5$	$\tau = 21$	$\tau = 63$	$\tau = 252$	$\tau = 5$	$\tau = 21$	$\tau = 63$	$\tau = 252$
Scenario 3.1: Trading data generated by the EEOW model of $\alpha = 0.3, \delta = 0.5, g = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \omega = \begin{pmatrix} 3 \\ 10 \end{pmatrix}, \Phi = \begin{pmatrix} 0.22 & 0.15 \\ -0.35 & 0.13 \end{pmatrix}, \Psi = \begin{pmatrix} 0.12 & 0.09 \\ 0.06 & 0.10 \end{pmatrix}, \begin{pmatrix} \alpha\mu_0 \\ 2\varepsilon_0 \end{pmatrix} = \begin{pmatrix} 8 \\ 16 \end{pmatrix}$								
HMM	0.0219	0.0206	0.0105	0.0083	0.0000	0.0000	0.0000	0.0000
	(0.0307)	(0.0256)	(0.0132)	(0.0109)	(0.0000)	(0.0000)	(0.0000)	(0.0000)
EHO	0.1845	0.0914	0.046	0.0152	0	0	0	0
	(0.2404)	(0.1188)	(0.058)	(0.0236)	(0)	(0)	(0)	(0)
DY	0.1809	0.0956	0.0531	0.0315	0.0371	0.0371	0.0371	0.0371
	(0.2385)	(0.118)	(0.0642)	(0.0407)	(0.0483)	(0.0483)	(0.0483)	(0.0483)
EEOW	0.1711	0.0761	0.0374	0.0086	0	0	0	0
	(0.2282)	(0.1005)	(0.0475)	(0.0111)	(0)	(0)	(0)	(0)
Scenario 3.2: Trading data generated by the EEOW model of $\alpha = 0.2, \delta = 0.4, g = \begin{pmatrix} 4E^{-5} \\ E^{-5} \end{pmatrix}, \omega = \begin{pmatrix} 4 \\ 12 \end{pmatrix}, \Phi = \begin{pmatrix} 0.33 & -0.1 \\ -0.02 & 0.01 \end{pmatrix}, \Psi = \begin{pmatrix} 0.14 & 0.12 \\ 0.18 & 0.18 \end{pmatrix}, \begin{pmatrix} \alpha\mu_0 \\ 2\varepsilon_0 \end{pmatrix} = \begin{pmatrix} 4 \\ 13 \end{pmatrix}$								
HMM	0.0187	0.0139	0.0087	0.0066	0.0000	0.0000	0.0000	0.0000
	(0.03)	(0.0183)	(0.0112)	(0.008)	(0.0000)	(0.0000)	(0.0000)	(0.0000)
EHO	0.1878	0.0878	0.0457	0.0183	0	0	0	0
	(0.2312)	(0.1089)	(0.061)	(0.0242)	(0)	(0)	(0)	(0)
DY	0.183	0.0841	0.0615	0.0415	0.0766	0.0766	0.0766	0.0766
	(0.2195)	(0.1086)	(0.0793)	(0.0601)	(0.0816)	(0.0816)	(0.0816)	(0.0816)
EEOW	0.1636	0.0778	0.0422	0.0093	0	0	0	0
	(0.2041)	(0.0991)	(0.0533)	(0.0113)	(0)	(0)	(0)	(0)
Scenario 3.3: Trading data generated by the EEOW model of $\alpha = 0.4, \delta = 0.6, g = \begin{pmatrix} E^{-5} \\ 0 \end{pmatrix}, \omega = \begin{pmatrix} 3 \\ 69 \end{pmatrix}, \Phi = \begin{pmatrix} -0.08 & 0.36 \\ 0.01 & -0.02 \end{pmatrix}, \Psi = \begin{pmatrix} 0.13 & 0.10 \\ -0.19 & 0.20 \end{pmatrix}, \begin{pmatrix} \alpha\mu_0 \\ 2\varepsilon_0 \end{pmatrix} = \begin{pmatrix} 2 \\ 69 \end{pmatrix}$								
HMM	0.017	0.0109	0.0051	0.0031	0.0000	0.0000	0.0000	0.0000
	(0.0238)	(0.0136)	(0.0065)	(0.0038)	(0.0000)	(0.0000)	(0.0000)	(0.0000)
EHO	0.1419	0.0857	0.0781	0.0766	0	0	0	0
	(0.1719)	(0.1001)	(0.0859)	(0.0768)	(0)	(0)	(0)	(0)
DY	0.1413	0.055	0.0287	0.0034	0.0003	0.0003	0.0003	0.0003
	(0.1763)	(0.0677)	(0.0382)	(0.0043)	(0.0021)	(0.0021)	(0.0021)	(0.0021)
EEOW	0.1261	0.0486	0.026	0.0057	0	0	0	0
	(0.1573)	(0.0599)	(0.0344)	(0.0072)	(0)	(0)	(0)	(0)
Scenario 3.4: Trading data generated by the EEOW model of $\alpha = 0.2, \delta = 0.55, g = \begin{pmatrix} 0 \\ 2E^{-5} \end{pmatrix}, \omega = \begin{pmatrix} 15 \\ 48 \end{pmatrix}, \Phi = \begin{pmatrix} -0.01 & 0.03 \\ -0.07 & 0.02 \end{pmatrix}, \Psi = \begin{pmatrix} 0.08 & 0.07 \\ 0.03 & 0.03 \end{pmatrix}, \begin{pmatrix} \alpha\mu_0 \\ 2\varepsilon_0 \end{pmatrix} = \begin{pmatrix} 6 \\ 40 \end{pmatrix}$								
HMM	0.0078	0.0092	0.0078	0.005	0.0000	0.0000	0.0000	0.0000
	(0.013)	(0.012)	(0.0104)	(0.0062)	(0.0000)	(0.0000)	(0.0000)	(0.0000)
EHO	0.1928	0.0867	0.0822	0.0693	0	0	0	0
	(0.2126)	(0.1118)	(0.097)	(0.08)	(0)	(0)	(0)	(0)
DY	0.1744	0.0635	0.0433	0.0097	0.01	0.01	0.01	0.01
	(0.2098)	(0.0847)	(0.0545)	(0.0126)	(0.014)	(0.014)	(0.014)	(0.014)
EEOW	0.1679	0.06	0.04	0.0071	0	0	0	0
	(0.2071)	(0.0803)	(0.0507)	(0.0099)	(0)	(0)	(0)	(0)

Table S-III-*Continued*

	$BIAS_{PIN_{[0,\tau]}}$ $\left(RMSE_{PIN_{[0,\tau]}}\right)$				$BIAS_{PSOS_{[0,\tau]}}$ $\left(RMSE_{PSOS_{[0,\tau]}}\right)$			
	$\tau = 5$	$\tau = 21$	$\tau = 63$	$\tau = 252$	$\tau = 5$	$\tau = 21$	$\tau = 63$	$\tau = 252$
Scenario 4.1: Trading data generated by extended EEOW model of $\alpha = 0.2, \delta = 0.5, \theta = \theta' = 0.2, \varepsilon_t = 50, g = \begin{pmatrix}0\\0\end{pmatrix}, \omega = \begin{pmatrix}12\\112\end{pmatrix}, \Phi = \begin{pmatrix}0.03 & 0.02\\0.04 & 0.28\end{pmatrix}, \Psi = \begin{pmatrix}0.08 & 0.06\\-0.01 & 0.01\end{pmatrix}, \begin{pmatrix}\alpha\mu_0\\2\varepsilon_0 + 2\theta\upsilon_0\end{pmatrix} = \begin{pmatrix}12\\113\end{pmatrix}$								
HMM	0.0163	0.0112	0.0061	0.003	0.0143	0.0095	0.0069	0.0069
	(0.0289)	(0.0148)	(0.0078)	(0.0039)	(0.0250)	(0.0205)	(0.0148)	(0.0131)
EHO	0.1354	0.1	0.091	0.0844	0.2353	0.2791	0.3076	0.3143
	(0.1662)	(0.1146)	(0.0983)	(0.0888)	(0.3196)	(0.3015)	(0.3126)	(0.3154)
DY	0.1115	0.0518	0.0249	0.0019	0.1912	0.0941	0.0407	0.0028
	(0.133)	(0.0623)	(0.0309)	(0.0025)	(0.2248)	(0.1171)	(0.0504)	(0.0036)
EEOW	0.1771	0.1458	0.1386	0.1315	0.2353	0.2791	0.3076	0.3143
	(0.2037)	(0.1591)	(0.1432)	(0.1335)	(0.3196)	(0.3015)	(0.3126)	(0.3154)
Scenario 4.2: Trading data generated by extended EEOW model of $\alpha = 0.3, \delta = 0.4, \theta = \theta' = 0.3, \varepsilon_t = 40, g = \begin{pmatrix}2E^{-5}\\E^{-5}\end{pmatrix}, \omega = \begin{pmatrix}40\\28\end{pmatrix}, \Phi = \begin{pmatrix}0.13 & -0.19\\-0.26 & 0.20\end{pmatrix}, \Psi = \begin{pmatrix}0.09 & 0.06\\0.23 & 0.3\end{pmatrix}, \begin{pmatrix}\alpha\mu_0\\2\varepsilon_0 + 2\theta\upsilon_0\end{pmatrix} = \begin{pmatrix}40\\60\end{pmatrix}$								
HMM	0.0205	0.0115	0.0069	0.0036	0.0164	0.0152	0.0143	0.0124
	(0.0284)	(0.0144)	(0.0084)	(0.0044)	(0.0185)	(0.0154)	(0.0135)	(0.0110)
EHO	0.1542	0.0729	0.0644	0.0711	0.1214	0.138	0.1364	0.1317
	(0.1881)	(0.086)	(0.0722)	(0.0724)	(0.1542)	(0.1447)	(0.139)	(0.1325)
DY	0.1462	0.054	0.0286	0.003	0.0821	0.0375	0.0237	0.0121
	(0.1795)	(0.0672)	(0.0365)	(0.004)	(0.0948)	(0.0464)	(0.0315)	(0.0132)
EEOW	0.1318	0.0534	0.0329	0.0159	0.1214	0.138	0.1364	0.1317
	(0.1674)	(0.0675)	(0.0417)	(0.0189)	(0.1542)	(0.1447)	(0.139)	(0.1325)
Scenario 4.3: Trading data generated by extended EEOW model of $\alpha = 0.4, \delta = 0.5, \theta = \theta' = 0.2, \upsilon_t = 40, g = \begin{pmatrix}E^{-5}\\E^{-5}\end{pmatrix}, \omega = \begin{pmatrix}18\\38\end{pmatrix}, \Phi = \begin{pmatrix}0.07 & -0.04\\-0.28 & 0.29\end{pmatrix}, \Psi = \begin{pmatrix}0.05 & 0.07\\0.25 & 0.22\end{pmatrix}, \begin{pmatrix}\alpha\mu_0\\2\varepsilon_0 + 2\theta\upsilon_0\end{pmatrix} = \begin{pmatrix}20\\40\end{pmatrix}$								
HMM	0.0269	0.0102	0.0078	0.0046	0.0320	0.0275	0.0222	0.0217
	(0.032)	(0.0124))	(0.0095)	(0.0057)	(0.0366)	(0.0287)	(0.0259)	(0.0214)
EHO	0.0949	0.0406	0.0271	0.025	0.1375	0.1573	0.155	0.16
	(0.1173)	(0.0492)	(0.0333)	(0.0267)	(0.1916)	(0.1685)	(0.158)	(0.1607)
DY	0.0974	0.0361	0.0209	0.0049	0.124	0.05	0.0268	0.0123
	(0.121)	(0.043)	(0.0254)	(0.0059)	(0.1392)	(0.0627)	(0.0328)	(0.015)
EEOW	0.1081	0.0662	0.0546	0.0565	0.1375	0.1573	0.155	0.16
	(0.1392)	(0.0779)	(0.0609)	(0.0572)	(0.1916)	(0.1685)	(0.158)	(0.1607)
Scenario 4.4: Trading data generated by extended EEOW model of $\alpha = 0.45, \delta = 0.6, \theta = \theta' = 0.15, \upsilon_t = 10, g = \begin{pmatrix}E^{-5}\\2E^{-5}\end{pmatrix}, \omega = \begin{pmatrix}2\\15\end{pmatrix}, \Phi = \begin{pmatrix}0.31 & -0.13\\-0.54 & 0\end{pmatrix}, \Psi = \begin{pmatrix}0.19 & 0.18\\0.26 & 0.22\end{pmatrix}, \begin{pmatrix}\alpha\mu_0\\2\varepsilon_0 + 2\theta\upsilon_0\end{pmatrix} = \begin{pmatrix}3\\16\end{pmatrix}$								
HMM	0.0249	0.0199	0.0137	0.0071	0.0350	0.0308	0.0249	0.0256
	(0.0209)	(0.0184)	(0.0116)	(0.0126)	(0.0328)	(0.0295)	(0.0282)	(0.0266)
EHO	0.1036	0.0612	0.0459	0.0368	0.0779	0.1226	0.1279	0.1284
	(0.1296)	(0.0777)	(0.056)	(0.0431)	(0.1391)	(0.1361)	(0.1316)	(0.1292)
DY	0.092	0.057	0.0404	0.0283	0.1334	0.0618	0.0396	0.0339
	(0.1152)	(0.0695)	(0.0499)	(0.032)	(0.1454)	(0.0751)	(0.0485)	(0.0398)
EEOW	0.1109	0.0546	0.0372	0.0291	0.0779	0.1226	0.1279	0.1284
	(0.137)	(0.0677)	(0.0458)	(0.0311)	(0.1391)	(0.1361)	(0.1316)	(0.1292)

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